Journal of Mathematical Psychology 108 (2022) 102653

Contents lists available at ScienceDirect

Journal of Mathematical Psychology

journal homepage: www.elsevier.com/locate/jmp

On reference dependence and complementary symmetry

Michał Lewandowski [a], Łukasz Woźny [b,*]

[a] Institute of Econometrics, Warsaw School of Economics, al. Niepodległości 162, 02-554 Warszawa, Poland
[b] Department of Quantitative Economics, Warsaw School of Economics, al. Niepodległości 162, 02-554 Warszawa, Poland

ARTICLE INFO

Article history:
Received 15 December 2020
Received in revised form 22 December 2021
Accepted 2 February 2022
Available online xxxx

Keywords:
Complementary symmetry
Buying price
Selling prices
Short-selling price
Insurance
Reference dependence

ABSTRACT

This paper reevaluates the complementary symmetry property and the corresponding experimental evidence. Originally the property was stated for binary risky prospects. We generalize it to arbitrary state-contingent real-valued outcomes, thus extending the domain of choice from risk to uncertainty/ambiguity and allowing for multiple outcomes. We consider various observable tasks related to the elicitation of buying and selling prices. In particular, for selected reference point models, we derive relevant definitions of gains and losses, and identify pairs of prices satisfying the complementary symmetry property. We then run an experiment to test these new predictions, and find that while some reference point models can be refuted based on our data, others perform reasonably well.

1. Introduction

A buying price (or a Willingness-To-Pay) of an uncertain prospect is the maximum amount of money the buyer is willing to pay for it. A selling price (or a Willingness-To-Accept) of a prospect is the minimum amount of money the seller is willing to accept for giving it up. There are several models of buying and selling prices. Under one approach formulated by Birnbaum and Zimmermann (1998, p. 178–180) and later applied both in economics (e.g. Schmidt, Starmer, & Sugden, 2008) as well as in finance (e.g. Carmona, 2008), the decisions to buy or sell are made based on the result of an adjustment of the consequences of the gambles to reflect buying or selling prices. For example, in the decision to sell a binary prospect paying x or y, where $x > y$, for a price of P, the seller "considers it a 'loss' of $x - P$ if the prospect might win x, since the seller gave up the opportunity to win; and if the prospect pays only y, the seller considers that a 'win' of $P - y$". Birnbaum, Yeary, Luce, and Zhao (2016, p.184–185). In their model, when we denote by $\sim$ a binary preference relation over prospects, the buying and the selling price of prospect l are two scalars $B(l)$ and $B^*(l)$ satisfying, respectively:

$$l - B(l) \sim 0, \tag{1}$$

$$B^*(l) - l \sim 0, \tag{2}$$

where $l - B(l)$ denotes a prospect in which $B(l)$ is subtracted from each prize in l and $B^*(l) - l$ is a prospect in which $B^*(l)$ is added to the negative of each prize in l.

Birnbaum and Zimmermann (1998) and Birnbaum et al. (2016) have shown that (1) and (2) combined with a popular parametric form of cumulative prospect theory imply the so called complementary symmetry (CS) property. The property states that $B(x, y; p) + B^*(x, y; 1 - p) = x + y$, where $(x, y; p)$ denotes a prospect paying x with probability p, or y with probability $1 - p$. Further work has shown (Chudziak, 2020; Lewandowski, 2018; Wakker, 2020), that the CS property follows from more general assumptions. In particular, Wakker (2020) proved that it is true under (1) and (2) for any binary relation $\sim$ on any subset of binary prospects, thus not even requiring utility representation. These results imply that the CS property is a property of the definitions (1) and (2) irrespectively of particular assumptions imposed on preferences.

The first contribution of this paper is to further generalize the CS property in two respects. We show that the property holds on a broader domain of decisions under uncertainty or ambiguity, i.e. for preferences defined over state-contingent outcomes. Hence no probability distribution needs to be assumed. Second, the original CS property is stated using a pair of complementary binary prospects. We show how to extend it to multiple outcome prospects, and propose the principle of constructing the complementary prospect using the notion of a perfect hedge.

In actuarial science there are two main principles of premium calculation: the *zero-utility*, and the *mean-value* principle. Scalars $B(l)$ and $B^*(l)$ are premiums calculated according to the zero-utility principle. The mean-value principle leads to the premiums $S(l)$ and $S^*(l)$ defined as:

$$S(l) \sim l, \tag{3}$$

$$-S^*(l) \sim -l. \tag{4}$$

* Corresponding author.
E-mail addresses: michal.lewandowski@sgh.waw.pl (M. Lewandowski), lukasz.wozny@sgh.waw.pl (Łu. Woźny).

https://doi.org/10.1016/j.jmp.2022.102653

In principle, one can propose an alternative CS property in which B and B^* are replaced by S and S^*, respectively. Our second contribution is to show that such property is not satisfied in general, only in the special case in which $S = B$ and $S^* = B^*$ (in which case it becomes the original CS property). This makes the CS property unique and hence good for testing the models in which the scalars B, B^*, S, S^* are related to some observable quantities.

Up till now we discussed complementary symmetry as the property of the pair of functionals B and B^* defined by (1), and (2), that holds irrespectively of any interpretation. In the framework of preferences defined over wealth changes, that are a special (albeit very popular in applications) case of more general reference-dependent preferences (see Sugden, 2003), we analyze few popular reference point models, and for each of them identify the appropriate definitions of buying and selling prices, and – as a consequence – a specific version of the CS hypothesis. Specifically, we follow the suggestion of Barberis (2013, p.178–179) who states that defining gains and losses for each decision context is the main challenge to apply any reference-dependent model and that the best way to tackle this problem is to derive the predictions of a reference-dependent theory "under a variety of plausible definitions of gains and losses, and to then test these predictions, both in the laboratory and in the field". This important issue of defining gains and losses was also well-recognized by the pioneers of the idea of reference-dependence (Kahneman & Tversky, 1979; Markowitz, 1952). In particular, we analyze three reference point models: the *SQ model* that sets the reference point at the status quo wealth, which is allowed to be random; the *IW model*, that sets the reference point at the initial wealth level, which is a nonrandom part of the status quo wealth, and the *MM model*, that sets the reference point at the security (maxmin) level of the alternatives in the choice set. Following Eisenberger and Weber (1995), in addition to the standard buying and selling prices, for each of the models, we also consider their short versions. For example, the selling-short price of prospect l is the minimum price the decision maker is willing to accept for taking a short position in l, i.e. accepting $-l$.[1] Our third contribution is to show how a specific formal representation, such as one of (1)–(4), of the buying (and the buying-short), the selling (and the selling-short) prices, arises as a result of imposing a specific reference point model (and thus defining gains and losses in a specific way). We show which of the six possible pairs out of four considered prices satisfy (a version of) the CS property according to each of the three reference point models.

One can use the CS property to test various reference point models. For example, the model proposed by Birnbaum and Zimmermann (1998, p.178–180), in which a buying price of prospect l is represented by $B(l)$ and the selling price of l is represented by $B^*(l)$, can be tested by asking whether the CS property is satisfied for the pair of elicited buying and selling prices. In fact it has been done for binary equal chance prospects by Birnbaum (2018) and Birnbaum and Zimmermann (1998) by reanalyzing experimental data of Birnbaum and Sutton (1992), and for general binary prospects by Birnbaum et al. (2016). In both data sets the sum of the median reported buying price of $(x, y; p)$ and the median reported selling price of $(x, y; 1-p)$ was always found to be below $x+y$. It was shown that, contrary to the theoretical prediction, the difference depends both on the outcome range, i.e. $|x - y|$ (being decreasing in it[2]) and on the value of p (inverse-U shape with a downward kink for $p = 0.5$).

One of our goals is to test complementary symmetries involving other pairs of prices as well (i.e. other than the buying and the selling price). We run two experiments. In the first one, conducted at Warsaw School of Economics, we tested binary equal chance prospects selected from the prospects used by Birnbaum and Sutton (1992). In addition to replicating their results, we found that while the data is rarely consistent with the CS hypothesis for the pair of elicited buying and selling prices, and it is more often consistent with the CS hypothesis for the pair of elicited buying and selling-short prices. In fact, we tested all six pairs of prices, and found that the CS hypothesis for the pair of elicited buying and selling-short prices is by far most consistent with the data, thus, as we show, refuting the SQ and the MM models. This is our fourth contribution.

Finally, in the second experiment conducted at the University of Virginia we tested the CS properties for more general prospects. In addition to binary equal chance prospects we elicited buying, selling and selling-short prices for an unequal chance binary prospect as well as an uncertain quaternary prospect. We found that while there is more noise in the data for the uncertain prospect, the main result remains the same, i.e. the individual data satisfy the CS hypothesis using a pair of elicited buying and selling-short prices more often than using a pair of elicited buying and selling prices. This is our fifth and final contribution.

2. The generalized complementary symmetry

Let N be a finite number of states of nature. Objects of choice are prospects — state contingent real-valued outcomes $l \in \mathbb{R}^N$. Let $\lambda \in \mathbb{R}$ denote the prospect $\lambda\mathbb{1}$, where $\mathbb{1}$ is a unit vector in $\mathbb{R}^N$. Preferences are given by a weak order, $\succcurlyeq$, over the set of prospects, with $\sim$ and $\succ$ denoting its symmetric and asymmetric part, respectively. The domain is very general: it allows for state-contingent preferences, lack of probabilistic sophistication, ambiguity aversion etc., it also includes decisions under risk with preferences defined over probability distributions.[3]

We define four functionals B, B^*, S, S^*, each of them mapping $\mathbb{R}^N$ to $\mathbb{R}$, such that (1), (2), (3), (4) holds for each prospect l. To assure existence and uniqueness[4] of $B(l), B^*(l), S(l), S^*(l)$ for each l, as is typically done, we assume that $\succcurlyeq$ is continuous and monotone.[5] Then, B, B^* as well as S, S^* are related by the following symmetry property, i.e. for any prospect l, $B^*(l) = -B(-l)$ and $S^*(l) = -S(-l)$. Moreover, both B and B^* satisfy the translation invariance property, i.e. for any prospect l and $\lambda \in \mathbb{R}$, $B(l+\lambda) = B(l) + \lambda$ (same for B^*), while S and S^* in general do not.[6] We say that a pair of prospects (l', l'') is a perfect hedge if it satisfies $l' + l'' = \theta$ for some scalar θ. Note that l' and l'' exhibit maximal negative correlation with each other: accepting a portfolio of l' and l'' removes uncertainty completely. We now state our first result.[7]

1 It can also be thought of as selling insurance against the risk $-l$. In finance, short selling refers to a sale of security not held by the seller at the time of transaction. Anticipating the fall of a security's price, the investor sells the given security and *later* buys it back on the *market* at a lower price. In our context neither the market, nor any kind of dynamics is involved. Following Eisenberger and Weber (1995) we use the term short-selling to denote the income position a subject is moving into when accepting $-l$.

2 For example, for $x = 48, y = 60, p = 0.5$ the sum of buying and selling prices was 104, being slightly below $x + y = 108$, while for $x = 12, y = 96, p = 0.5$, the sum dropped to 75, significantly below $x + y = 108$.

3 To see this suppose π is a probability measure over $\mathbb{R}^N$. The (induced) probability distribution of a prospect l is then $p_l : \mathbb{R} \to [0, 1]$, where $p_l(x) = \sum_{i \in N: l(i)=x} \pi(i)$ for any $x \in \mathbb{R}$. Decisions are under conditions of risk if any two prospects that induce the same probability distribution are equally preferred.

4 Generalizations including nonuniqueness are possible. See Wakker (2020) for relevant results and ideas that can be extended to prospects in $\mathbb{R}^N$.

5 That is: for any prospect l the sets $\{\lambda \in \mathbb{R} : \lambda \succcurlyeq l\}$ and $\{\lambda \in \mathbb{R} : \lambda \preccurlyeq l\}$ are closed and that $l(i) > l'(i)$ for all $i \in N$ implies $l \succ l'$.

6 Both properties follow from similar reasoning as in Chudziak (2020).

7 Observe that continuity and monotonicity of $\succcurlyeq$ is not required for Proposition 1. See also Wakker (2020).

Proposition 1. Let (l', l'') be a perfect hedge. The following holds:

$$B(l') + B^*(l'') = \theta. \tag{5}$$

Proof. Using the symmetry property of B and B^* and the translation invariance of B^*, we have $0 = B(l') - B(l') = B(l') + B^*(-l') = B(l') + B^*[\theta - l'] - \theta$, thus obtaining (5). □

For any prospect l and any scalars λ, θ the pair of prospects (l', l'') with $l' := l + \lambda$ and $l'' := \theta - \lambda - l$, respectively, is a perfect hedge. This observation allows us to state our main result on the generalized complementary symmetry. □

Corollary 1. *Let l be a prospect. The following holds for any scalars λ, θ:*

$$B(l + \lambda) + B^*(\theta - \lambda - l) = \theta. \tag{6}$$

Proof. Note that a pair of prospects $(l + \lambda, \theta - \lambda - l)$ constitute a perfect hedge for any scalars θ, λ and hence the result is true by a direct application of Proposition 1. □

This result is important for few reasons. First, it generalizes the previous results on the CS from probability distributions of binary prospects to prospects over $\mathbb{R}^N$. Indeed, for $N = 2$ taking for example $l = (x, y)$ and letting $\theta = x + y$ with $\lambda = 0$ we obtain the standard CS. Second, the corollary shows that the complementary symmetry property follows immediately from general properties of translation invariance and symmetry. Third, as it is clear from the construction, complementary symmetry does not depend on the (existence of) underlying probability distribution. This shows CS is true for decisions under uncertainty or ambiguity.

Note that property (6) is true irrespective of the value of θ or/and λ. This gives some degree of freedom in choosing them so that the constructed prospects l' and l'' have additionally some desirable properties. For example, one may want to choose θ and λ so that $l' = l + \lambda$ and $l'' = \theta - l - \lambda$ are both gain prospects and their ranges coincide.[8] This is the case if and only if[9] $\lambda \geq -\min(l)$ and $\theta = \min(l) + \max(l) + 2\lambda$. Therefore, if $l = (x, y)$ with $0 \leq x < y$, then taking $\lambda = 0$ we get:

$$\theta = \min(l) + \max(l) = x + y,$$
$$l' = (x, y), l'' = (y, x).$$

precisely the choice of prospects used by Birnbaum and Zimmermann (1998) and Birnbaum et al. (2016). In the more general case, for example if $l = (-40, -20, 0, 80)$, setting $\lambda = 120$ and $\theta = -40 + 80 + 240 = 280$ gives $l' = (80, 100, 120, 200)$ and $l'' = (200, 180, 160, 80)$.

We say that a pair of price functionals (P, P^*), each mapping $\mathbb{R}^N$ to $\mathbb{R}$, satisfies CS if (6) is true for all prospects l and all scalars λ, θ, with P, P^* replacing B, B^*. By Corollary 1 we know that (B, B^*) satisfies CS. Since the construction of S and S^* is similar to that of B and B^*, it is natural to ask whether (S, S^*) also satisfies CS. It is summarized in the next proposition.

Proposition 2. The following three statements are equivalent:

i. S is translation invariant,
ii. $S(l) = B(l)$ for all prospects l,
iii. $S(l + \lambda) + S^*(\theta - \lambda - l) = \theta$ holds for all prospects l and scalars θ, λ.

Proof. (i.⇒ii) Take any prospect l. By definition of S and B, $S(l - B(l)) \sim l - B(l) \sim 0$. From monotonicity $S(l - B(l)) = 0$. By translation invariance, $S(l - B(l)) = S(l) - B(l)$. Hence $S(l) = B(l)$.
(ii.⇒iii.) Since $S^*(l) = -S(-l)$, $B^*(l) = -B(-l)$, so $S(l) = B(l)$ implies $S^*(l) = B^*(l)$. This is true for all l. So iii. must be true by (6).
(iii.⇒i.) Take any prospect l and $\theta \in \mathbb{R}$. We must prove that $S(l + \theta) = S(l) + \theta$. Define $l' := l - \lambda + \theta$ for some $\lambda \in \mathbb{R}$. So $S(l + \theta) = S(l' + \lambda) = \theta - S^*(\theta - \lambda - l') = S(l' + \lambda - \theta) + \theta = S(l) + \theta$, where the second equality follows from iii., the third from $S^*(l) = -S(-l)$ for all l and the first and the last from the definition of l'. □

As shown, the only case when (S, S^*) satisfies CS is when S (and S^*) is identical to B (and B^*, respectively), in which case the corresponding CS property reduces to (6).

3. Reference point models and their testable predictions

In this section, we consider three reference point models and, within them, we analyze how buying and selling decisions define buying and selling prices corresponding to the mathematical quantities B, B^*, S, S^* defined by (1)–(4). Following Eisenberger and Weber (1995) we consider four prices, i.e. willingness-to-pay and willingness-to-accept both in the positive as well as in the negative frame.

An uncertain prospect l with nonnegative prizes promises to pay its owner the amount l_i after Nature chooses (i.e. the uncertainty is resolved) a single state $i \in N$. Having a long position in a prospect is the same as *being promised to be paid l_i*, whereas having a short position is a *promise to pay l_i*.[10] We thus consider the maximum price the decision maker is **willing to pay**:

i. for (taking a long position in) l, called the buying price of l and denoted by $b(l)$,
ii. to cancel a short position in l, called the buying-short price of l and denoted by $bs(l)$.

We similarly consider the minimum price the decision maker is **willing to accept**:

i. for giving up (a long position in) l, called the selling price of l and denoted by $s(l)$,
ii. for taking a short position in l, called the selling-short price of l and denoted by $ss(l)$.[11]

We model reference-dependence following the approach of Sugden (2003). This approach, also used in Schmidt et al. (2008), models reference dependence in a state-contingent way defining the reference point and hence gains and losses state-by-state.[12]

Even though mathematically indistinguishable, we differentiate between prospects and wealth (Savage) acts.[13] The latter are

8 Such properties could be justified on the grounds of framing effects literature (e.g. Sayman & Öncüler, 2005). In particular, it was demonstrated that the relative attractiveness of a prospect may vary depending on the choice task being presented either in the loss or in the gain frame (Irwin, 1994; McClelland & Schulze, 1991). Similarly, it was argued that valuation of the prospect may depend on its range (Kontek & Lewandowski, 2018; Mellers, Ordoñez, & Birnbaum, 1992). Here, in testing CS property, one may use particular choices of λ and θ to control for these two effects.

9 We want to thank the anonymous Referee for suggesting this statement and the example below. Here, $\min(l)$ and $\max(l)$ denote, respectively, the minimum and the maximum payoff in prospect l.

10 Alternatively, one can think of selling-short and buying-short as, respectively, selling and buying insurance against the risk of $-l$.

11 Whenever it is clear from the context we drop the argument and simply write b, s, bs, ss.

12 Kőszegi and Rabin (2006, 2007) suggest an alternative way in which state-dependence is ignored. In their approach preferences are over probability distributions, and the reference point is assumed to be stochastically independent of the stochastic outcome. Schmidt et al. (2008) have shown that this approach leads to paradoxical implications when random reference points are allowed.

13 The term act is used as in Savage (1954) whose choice objects are mathematically identical to ours.

Table 1
Wealth levels for Accept and Reject decisions and the corresponding reference points for given tasks with the initial wealth W, prospect l and the price P.

Tasks	Choice alternatives		Reference point models		
	Accept	Reject (status quo)	RSQ	IW	MM
Buying	$W+l-P$	W	W	W	W
Selling	$W+P$	$W+l$	$W+l$	W	$W+P$
Selling-short	$W-l+P$	W	W	W	W
Buying-short	$W-P$	$W-l$	$W-l$	W	$W-P$

denoted by f, g, h, and have consequences interpreted as levels of wealth, while the former are derived from the latter with consequences being changes of wealth levels relative to a given reference wealth act. We assume a binary relation $\succcurlyeq$ over the set of prospects and a ternary relation over wealth acts, where $f \succcurlyeq_h g$ means that f is weakly preferred to g viewed from the perspective of h, the reference act. To formally express the idea that decision makers care about changes in wealth[14], we assume that for any reference act h, $\succcurlyeq_h$ is derived from $\succcurlyeq$ by taking $f \succcurlyeq_h g \iff f-h \succcurlyeq g-h$.[15]

When modeling buying and selling prices we consider a prospect l that is being sold or bought, with prizes denoting income (money won or lost). We assume there is no other source of uncertainty except for l, i.e. all prior uncertainty has been resolved at the time of decision, and there is no background risk. Thus the decision maker's initial wealth $W \in \mathbb{R}$ is a constant (i.e. deterministic) wealth act.

We now consider three popular reference point models, that are empirically grounded but also relevant for our discussion on the CS property. First, the *Status Quo* (SQ) model sets the reference point at the status quo wealth position. It is based on the idea underlying the model of buying and selling prices of Birnbaum and Zimmermann (1998, p. 178–180) and of Schmidt et al. (2008), which extends the original prospect theory (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992) by allowing the reference point to be random. Second, the *initial wealth* (IW) level model sets the reference point at the initial wealth level. This models differs from the SQ model in that it takes only the deterministic part of the status quo wealth. For example, in the case of selling prospect l the status quo wealth contains the deterministic part W and the random part l. The SQ model sets the reference point at $W+l$ whereas the IW model at W. Note that the IW model is consistent with the original prospect theory as well as with the expected utility model of wealth in which the utility function exhibits Constant Absolute Risk Aversion and thus excludes wealth effects. Third and finally, the *maxmin* reference point model (MM) sets the reference point at the maximum of security levels of each choice alternatives in the choice set. This model has been proposed by Hershey and Schoemaker (1985) and was then generalized by Schneider and Day (2018).[16]

Table 1 summarizes the relevant wealth acts for all four tasks (buying, selling, selling-short and buying-short) and the three reference point models. Table 2 gives the resulting wealth changes,

[14] This idea commonly underlies most applications of a reference-dependent model (Schmidt et al., 2008; Sugden, 2003). Yet, some authors recognize that a simple subtraction of wealth levels may not reflect well the psychology of reference dependence. In particular, Luce (2000) postulates using a joint receipt operation that generalizes addition (and consequently subtraction).

[15] The assumption needed for this to be satisfied is $f \succcurlyeq_h g \iff f+(h'-h) \succcurlyeq_{h'} g+(h'-h)$ for all f, g, h, h'.

[16] The counterpart of this model is the minmax reference point model, analyzed by Baillon, Bleichrodt, and Spinu (2020), which takes the minimum over maximum prizes of each choice alternatives in the choice set. Since all our choice sets consist of two alternatives (buying vs. not buying, selling vs. not selling), and one of these two alternatives is deterministic with a prize in between the minimum and the maximum prize of the other alternative, the maxmin and the minmax models coincide in our case.

Table 2
Prospects for Accept and Reject decisions under various reference point models for the four analyzed tasks.

Tasks	RSQ		IW		MM	
	Accept	Reject	Accept	Reject	Accept	Reject
Buying	$l-P$	0	$l-P$	0	$l-P$	0
Selling	$P-l$	0	P	l	0	$l-P$
Selling-short	$P-l$	0	$P-l$	0	$P-l$	0
Buying-short	$l-P$	0	$-P$	$-l$	0	$P-l$

Table 3
Prices representations by the formulas (1)–(4) under the three reference point models.

Prices	SQ	IW	MM
Buying	B	B	B
Selling	B^*	S	B
Selling-short	B^*	B^*	B^*
Buying-short	B	S^*	B^*

and Table 3 the corresponding formal representations using the functionals defined by (1)–(4). Observe that there is a natural asymmetry in the status quo wealth position in the buying and selling tasks. The seller initially owns the prospect while the buyer does not. Under the SQ model this difference is irrelevant, only the net change of wealth positions matter. This leads to the selling price being represented by B^* defined by (2). Under the two other models this is not true, for example, under the IW model the selling price is represented by S defined by (3).

Note that the buying and selling-short prices correspond to, respectively, B, and B^*, in each of the three reference point models. The selling and buying-short prices, in contrast, each have different formal representations under the three models. Interestingly, the SQ model postulated by Schmidt et al. (2008) predicts that long and short prices are the same, whereas the MM model postulated by Schneider and Day (2018) predicts that buying and selling prices are the same. We summarize this fact below for further reference:

Proposition 3 (*Price Predictions*)**.** The following relationships between prices follow from the three models:

$$\begin{aligned} &SQ: && b = bs, \quad s = ss, \\ &MM: && b = s, \quad bs = ss, \\ &IW: && \textit{all different in general}. \end{aligned}$$

These findings are relevant for the CS property defined in Section 2. Indeed, consider the prices $b(l), s(l), bs(l), ss(l)$ elicited from a single individual. We say that a pair of elicited prices (p, p^*) satisfies the CS property if (6) holds with p and p^* replacing B and B^*, i.e. the following holds for any pair of scalars λ, θ:

$$p(l+\lambda) + p^*(\theta - \lambda - l) = \theta. \tag{7}$$

We now summarize the CS predictions, that follow from Table 3, Corollary 1, and Proposition 2.

Proposition 4 (*CS Predictions*)**.** The following pairs of prices satisfy the CS property in the three models:

$$\begin{aligned} &SQ: && (b, ss), (b, s), (bs, s), (bs, ss), \\ &MM: && (b, ss), (b, bs), (s, ss), (s, bs), \\ &IW: && (b, ss). \end{aligned}$$

Note that the only pair of prices that satisfies the CS under all three models is (b, ss). The symmetry for a pair of prices (b, s) tested by Birnbaum et al. (2016), as well as the symmetry for a pair (bs, ss), is implied only by the SQ model. The symmetries for a pair of prices (b, bs) and (s, ss) are implied only by the MM model and the symmetry for a pair (bs, s) is implied by both, the SQ and the MM model (but not the IW model). In the next section, we thus use these CS predictions in order to test the three reference point models.

4. Method

There were two experimental sessions, hereafter referred to as Experiment 1 and Experiment 2.

4.1. Instructions and stimuli

In Experiment 1 subjects were instructed to specify the buying, selling, selling-short and buying-short prices for selected prospects. In Experiment 2 they were only asked to specify the buying, selling and selling-short prices. Phrasing of the tasks was the same in both experiments (see the Appendix), only the mode for providing the answer differed: In Experiment 1 subjects were asked to give an integer amount with no restriction on the possible value range. In Experiment 2 subjects had to locate the price using a slider bar between the lowest and the highest prize with a grid of $1, no custom start position and the option to tick a "no value in this range" button. No direct incentives were provided to the subjects. In Experiment 2 subjects were paid for participation: they entered into a raffle of three Amazon gift cards worth $25.

4.2. Design and procedure

There were two experimental sessions, both online. Each subject received a link to participate and could fill in the survey or quit any time before the deadline. There was one trial for each person. The same subjects served in the between- and within-subject (as applied to tasks, not different gambles) parts of the design.

In both experiments the elicitation tasks were presented in random order. It took on average 10 min to complete Experiment 1 and 12 to complete Experiment 2.

Experiment 1

Each subject was asked to evaluate a single binary equal chance prospect described by means of a symmetric coin toss selected at random from four available ones with the following payoffs (60, 48), (72, 36), (84, 24), (96, 12) (all prizes denoting dollar amounts). These payoffs were taken from Birnbaum and Sutton (1992) for comparability reasons. Note that the sum of payoffs is identical for each four of them and equals 108.

Experiment 2

Each subject had to evaluate the buying price for a single prospect l selected (at random) from three available ones and the selling and selling-short prices for a corresponding complementary prospect $\max(l) + \min(l) - l$ (i.e. generated according to the procedure described on Section 2). The three prospects were:

- 2-eq: Equal chance of winning 84 or 24 dollars. The prospect was described by means of a symmetric coin toss.
- 2-ueq: 2 : 1 chance of winning 84 vs. 24 dollars. The prospect was described by means of a symmetric dice roll with 1, 3, 4, 6 rolled representing 2/3 probability and 2,5 rolled representing the remaining 1/3.
- 4-unc: An uncertain prospect of winning (200, 120, 100, 80). The prospect was described by means of a draw of one ball from an urn containing 100 white, black, red and green balls of unknown proportions, where each color designates one of the four amounts.

Table 4
Median buying, selling, buying-short and selling-short prices.

Ranges	96–12	84–24	72–36	60–48
b	20	49	39	48
s	50	54	54	54
bs	25	53	54	54
ss	80	57	66	60
Number of subjects	11	6	10	11

Note that 2-eq is identical to its complementary prospect whereas 2-ueq and 4-unc are not. Hence, in Experiment 2, our design allows for testing two of the six CS symmetries for all three prospects, namely the CS for two pairs of prices: (b, s) and (b, ss).

4.3. Subjects

Experiment 1

Participants were 49 undergraduate and graduate students from Warsaw School of Economics. We excluded these questionnaires (11 in total) that contained at least one (out of four given by the subject) price either smaller than the lower, or larger than the upper prospect prize.[17] The number of complete (nonexcluded) questionnaires for the four gambles were, respectively: 11, 6, 10, 11.

Experiment 2

Participants were 44 undergraduate students with no extensive training in decision theory from the University of Virginia Darden School of Business subject pool. We excluded all questionnaires (10 in total) that contained at least one answer: "no value in this range". The number of nonexcluded questionnaires for the three considered prospects were: 13, 11, 10, respectively.

5. Results

5.1. Experiment 1: Testing the CS for equal chance binary prospects

Table 4 presents median prices for the four considered lotteries. First, in agreement with a vast literature on WTA/WTP disparity (e.g. Birnbaum & Stegner, 1979; Birnbaum & Sutton, 1992; Schmidt & Traub, 2009; Viscusi & Huber, 2012) for a given lottery the selling price systematically exceeds the buying price. The difference is also substantial in magnitude: note that the median buying price is close to the lower lottery prize[18], whereas the median selling price is close to lottery mean values. Second, while the selling prices are close to lottery mean values, they rarely (in 1 out of 38 cases) exceed them.[19] The selling-short prices, on the contrary, are typically above lottery mean values (there were only 4 out of 38 cases, where it was not the case).

We report the results of a between-subject design to compare them to Birnbaum (2018) who also used such data from the study of Birnbaum and Sutton (1992). The results involving median prices are summarized in the right panel of Table 5 and show that the CS tested with the use of buying and selling-short prices is approximately[20] satisfied in all ranges (in fact exactly satisfied

17 Such answer may suggest error, lack of focus, or preference for losing money.

18 Except for the case $84 - 24$ which may be caused by a smaller sample size.

19 This finding confirms that people are generally risk averse towards equal-chance gain gambles (see for example Tversky & Kahneman, 1992, Table 3.).

20 The elicitation using indifference prices is likely to be noisy (Hey, Morone, & Schmidt, 2009). In addition to testing the corresponding equalities exactly, we also test them approximately. Let $q(l), q'(l)$ denote two sums of prices for prospect l: for example $q(l) = b(l) + ss(l)$ and $q'(l) = b(l) + s(l)$. We say $q(l), q'(l)$ are "equal" whenever $\frac{|q(l)-q'(l)|}{\max(l)-\min(l)} \leq \epsilon$ for some ϵ. In the exact version we take $\epsilon = 0$, in the approximate version we take $\epsilon = 10\%$.

Table 5
Sum of median prices and the number of subjects consistent with each of the six CS properties (approximately and exactly).

	Sum of median prices				No of subjects	
	96–12	84–24	72–36	60–48	approx.	Exactly
(b, s)	73	101	99.5	108	18	9
(b, ss)	103	108	108	108	30	15
(s, ss)	120	110	120	110	15	9
(bs, s)	85	105	105	108	16	11
(bs, ss)	108	110	112	108	20	9
(b, bs)	45	99	97.5	107	14	5

in three out of four). In contrast, the CS tested with the use of buying and selling prices is violated. Observe, our results also replicate (Birnbaum, 2018) finding: the sum of median buying and selling prices is below $x + y$ for each x, y and decreases with $|y - x|$. Note, however, that the same is not true if selling price is replaced by a selling-short price, in which case the sum is relatively stable at $x + y$ for each x, y. Among other symmetries only (bs, ss) is approximately satisfied in all prospect ranges (and exactly satisfied in two of them). The remaining symmetries are satisfied less often with (b, bs) behaving similarly to (b, s) (also decreasing in the range).

The within-subject design allows for testing the predictions of each of the three considered models for each individual subject. Interestingly, observe that CS properties provide a stronger test than equality of prices. Specifically, for example subject number 34, who evaluated prospect $(96 - 12)$, reported $(b, s, ss, bs) = (30, 70, 70, 30)$. It is easy to see that $b = bs$ and $s = ss$, which is true under the SQ model, yet none of the CS properties predicted by the SQ model is satisfied: $b + ss = b + s = bs + s = bs + ss = 100 \neq 108$.[21] The right panel of Table 5 shows how many subjects satisfy each of the six symmetries. Note that (b, ss) is by far the most frequently satisfied symmetry (30 out of 38 subjects, or almost 80% satisfied it approximately and 15 exactly). (b, s) by comparison is only satisfied for 18 subjects approximately and 9 exactly.[22] The symmetry that is least often satisfied is (b, bs).

Based on the symmetries predicted by each of the three reference point models, we classified all observations into these consistent with: all three models, IW and SQ, IW and MM, IW only and no model.[23] Table 6 shows that out of 38 subjects 20 were approximately consistent only with the IW model, as compared to 2 consistent with the IW and SQ and 1 consistent with the IW and MM models. If we demand exact consistency, then the numbers drop in favor of the 'no model' cluster, but remain at similar levels in the relative terms: 8 subjects consistent with the IW as compared to 1 subject for each: IW& MM and IW & SQ clusters. These results confirm that the SQ and the MM models are in general refuted by the data.

5.2. Experiment 2: Testing the CS properties for general prospects

Table 7 presents the median prices for the three types of prospects. In this experiment we test the CS for two pairs of prospects: (b, s) and (b, ss). The CS property requires that the sum equals 108 in the case of 2-eq and 2-ueq prospects and 280 in the case of 4-unc prospect. What we find is that the pair (b, ss) is closer to satisfying the CS property on the median level in all three prospect types (the sums equal 99.1, 99.2, and 262.55, respectively) than the pair (b, s) (for which the sums equal 76, 65.3, and 254, respectively. Yet the effect is stronger in the case of binary risky prospects than in the case of an uncertain quaternary prospect.

We then do the within-subject analysis. Table 8 shows how many subjects were consistent (approximately and exactly) with each of the three considered symmetries for the 2-eq prospect and the (b, s) and (b, ss) symmetries for the 2-ueq, and 4-unc prospects. What we find is that the (b, ss) symmetry was the most frequent both in approximate as well as exact terms. In the case of the 2-eq prospect, there were 5 out of 13 subjects approximately consistent (and 4 exactly) with the CS for the pair (b, ss) as compared to no subjects (neither approximately nor exactly) consistent with the CS for the pair (b, s). In the case of the 2-ueq prospect, the numbers were 4/3) as compared to 0/0 and in the case of 4-unc – 4/3 as compared to 2/0. The results suggest that subjects are more consistent with the CS property for (b, ss) than for (b, s). Yet, as our sample is small, it would be worth repeating the experiment to see if the results are robust.

Table 6
Validity of the CS in a within-subject comparison. Clusters group subjects according to consistency with a set of models out of the three considered.

Cluster	no of subjects	
	approx	Exactly
All models	7	4
IW & SQ	2	1
IW & MM	1	1
Only IW	20	8
No model	8	24

Table 7
Median buying price of a prospect (denoted by l') and median selling, and selling-short of a complementary prospect (denoted by l'') for three types of prospects.

	Median prices			CS for median prices	
	$b(l')$	$s(l'')$	$ss(l'')$	(b, s)	(b, ss)
2-eq	29.1	46.9	70	76	99.1
2-ueq	25.2	40.1	74	65.3	99.2
4-unc	97.55	156.45	165	254	262.55

Table 8
Number of subjects consistent (exactly and approximately) with each of the two CS symmetries for binary equal chance prospect (2-eq), binary unequal chance prospect (2-ueq), quaternary uncertain prospect (4-unc).

	Approximate			Exact		
	2-eq	2-ueq	4-unc	2-eq	2-ueq	4-unc
(b, s)	0	0	2	0	0	0
(b, ss)	5	4	4	4	3	3
no.	13	11	10	13	11	10

5.3. Discussion and alternative theories

Birnbaum et al. (2016) mention three groups of models for buying and selling prices: prospect theory loss aversion models (e.g. Birnbaum & Zimmermann, 1998; Schmidt et al., 2008), joint receipt models (e.g. Luce, 1991, 2000), and configural weight models (e.g. Bimbaum, 1982; Birnbaum, Coffey, Mellers, & Weiss, 1992; Birnbaum & Stegner, 1979).

Our analysis falls into the first group, yet it is more general as it concerns any model of wealth changes, not only prospect theory either in its original (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992), or its extended version (Schmidt et al., 2008).

Instead of wealth changes defined by a standard subtraction of a reference act from an evaluated act, the joint receipt models of Luce (2000) assume a generalized subtraction operation,

21 Note that $b + ss = b + s = bs + s = bs + ss$ implies $s = ss$ and $b = bs$, yet the opposite is not true.

22 The difference is even more pronounced if we exclude all subjects (7 approximately and 4 exactly) that satisfied all symmetries at once by reporting four identical prices.

23 Note that other possibilities are not possible, as the implications of the SQ or the MM models are contained in the implications of the IW model.

reflecting the psychology of joint receipt of a gain and a loss. Since this is a more general model than ours[24], it is capable of producing predictions consistent with our data, yet without imposing more restrictions on its structure, it does not commit to any particular interpretation and thus is too general to produce strong testable predictions such as the ones produced by the three analyzed reference point models.

Note that the asymmetry between the buyer and the seller reveals itself in at least two ways. First, there is a difference in initial positions: the seller initially owns the object being sold while the buyer does not. This impacts the definition of gains and losses in each of these two tasks and hence leads to the difference in buying and selling prices. Moreover, in a state-contingent model as the one analyzed here, if the status quo wealth is random, various dependencies may arise between the evaluated prospect and the status quo wealth; these dependencies matter for the difference in buying and selling prices. These types of effects are observed in the models analyzed in this paper. Yet the asymmetry in buying and selling goes beyond the difference in initial positions and the related effects. It is also reflected in the buyer's interest to pay *the smallest* possible price and the seller receiving *the highest* possible price. In the framework analyzed here, where preferences are precise (i.e. assumed to be complete, continuous, and monotonic, thus leading to unique indifference prices) this latter type of asymmetry does not matter, as there is a single price at which the decision maker switches from buying to not buying or selling to not selling.

The configural weight models, in contrast, assume that the decision maker may value the same object (here a prospect) differently depending on the "point of view". Specifically, "buyers would place greater weight on the lower ranked estimates of value than do sellers" (Birnbaum et al., 2016, p. 187). In Birnbaum and Stegner (1979) theory it would be argued that selling-short and buying-short tasks are simply other "points of view", which induce different values of the configural weighting parameter, omega, so we expect that buying, selling, selling-short, and buying-short prices might all be different, similarly as in the IW model analyzed here. Yet, note that in order to explain why our data is consistent with the CS property for the pair of prices (b, ss) more often than it is for the other price pairs, one would need to impose additional restrictions on the configural weight parameter; these restrictions would need to follow from plausible premises reflecting the psychology of different tasks.

There are also buying and selling price models based on preference imprecision (Dubourg, Jones-Lee, & Loomes, 1994; Giraud, 2010). According to these models, WTA or WTP are not treated as point estimates, but rather as "personal confidence intervals" within which their values lay, reflecting people's lack of perfect information about the object being traded and/or lack of full knowledge concerning one's preferences. When forced to provide point estimates, people are likely to give lower ranked estimates within the interval of WTPs according to the buyer's point of view, and higher ranked estimates within the interval of WTA's according to the seller's point of view. Thus these models are compatible with configural weight models and, as them, they are capable of capturing the latter kind of asymmetry between buying and selling.

[24] One could test whether our model is a good approximation of subjects' preferences by testing the axiom stated in footnote. Since the axiom implies a kind of invariance to the reference act (what matters is only the difference with respect to it), its strength and the type of restriction it implies depend on a particular reference point model, which determines the allowed reference acts, that the preferences are invariant to.

Future research should compare the two approaches, and test which kind of asymmetry is more relevant empirically.[25] Obviously the results will have implications for the CS property tested for different pairs of prices.

Acknowledgments

We want to thank Michael Birnbaum, Jacek Chudziak and Peter Wakker for their helpful suggestions on the previous draft of this paper. Special thanks to Joanna Sokołowska-Pohorille and Katarzyna Idzikowska for their help in organizing Experiment 1 and to Manel Baucells and the stuff at the Behavioral Research Lab at the University of Virginia (Emily El Sanadi and Aglio Rhys in particular), for their help in preparing and organizing Experiment 2. All errors remain ours. Łukasz Woźny acknowledges financial support by the National Science Center, Poland: NCN grant number UMO-2019/35/B/HS4/00346.

Appendix. Description of the four tasks in Experiments 1 and 2

Below we present the exact phrasing of the tasks for the special case of the prospect giving equal chance of winning 84 vs. 24 dollars. The phrasing for other prospects was analogous.

A.1. Buying price

Consider a chance to win a monetary prize whose value will be determined by a toss of a symmetric coin:

- Receive \$84, if it lands heads,
- Receive \$24, if it lands tails.

You can either walk away and receive nothing or pay for the toss and get a prize.

What is the maximal amount of money that you would pay for the toss?

Your answers in dollars:

Questions below may help you make sure you have set the right amount:

1. Think of some amount such that if that was the price you would definitely pay.
2. Now think of a slightly higher price. Would you still pay?
3. Keep increasing the prices and stop whenever you feel that paying the amount would still be ok for you, but paying any more would be too much.

A.2. Selling price

Consider a chance to win a monetary prize whose value will be determined by a toss of a symmetric coin:

- Receive \$84, if it lands tails,
- Receive \$24, if it lands heads.

[25] For example, in addition to the standard way of eliciting WTA and WTP one could ask for the *lowest* price at which the decision maker will not buy, or the *highest* price at which the decision maker *will not sell* a prospect. Note that according to the model analyzed here the standard WTP, WTA will be very close (infinitesimally, in mathematical terms), respectively, to the first and to the second of these additional prices. In contrast, models in which price estimates are not point-estimates predict that WTP and the first of these additional prices, or WTA and the second of these additional prices may lie further apart.

You have the right to toss the coin for free. Alternatively, you can give it up in return for a sure amount of money.

What is the minimal amount at which you would agree to give up the toss?

Your answer in dollars:

Questions below may help you make sure you have set the right amount:

1. Think of some offered amount at which you would definitely agree to give up.
2. Think of a slightly smaller amount. Would you still agree to give up?
3. Keep decreasing the prices and stop at the amount such that receiving it in return for the toss would still be ok for you, but any smaller amount would not be enough.

A.3. Selling-short price

Imagine that you face the risk of losing money. The value of your loss will be determined by a toss of a symmetric coin:

- Loss of $84, if it lands tails,
- Loss of $24, if it lands heads.

You can either opt out and walk away or accept the above risk in return for some sure amount of compensation: your payoff will be the compensation amount minus the loss amount that will be determined by the toss.

What is the minimal amount of compensation such that you would agree to take on this risk?

Your answers in dollars:

Questions below may help you make sure you have set the right amount:

1. Think of some amount at which you will definitely agree.
2. Think of a slightly smaller amount. Would you still agree?
3. Keep decreasing the compensation and stop at the amount which would still be ok for you, but any smaller amount would not be enough.

A.4. Buying-short price

Imagine that you face risk of losing money. The value of your loss will be determined by a toss of a symmetric coin:

- Loss of $84, if it lands heads,
- Loss of $24, if it lands tails.

Imagine that your choice is between accepting this risk or selling it to a third party. You cannot opt out, i.e. you need to choose one of these two options. If you choose to sell the risk, you need to pay a sure amount of money as a premium for risk.

What is the maximal premium amount you would pay to avoid the risk?

Your answer in dollars:

Questions below may help you make sure you have set the right amount:

1. Think of some amount such that if that was the premium you would definitely pay to avoid the risk.
2. Think of a slightly higher amount. Would you still pay?
3. Keep increasing the premium amounts and stop whenever you feel that paying that amount would still be ok, but paying any more would be too much and you would rather prefer to stay with the risk.

References

Baillon, A., Bleichrodt, H., & Spinu, V. (2020). Searching for the reference point. *Management Science*, *66*(1), 93–112.

Barberis, N. C. (2013). Thirty years of prospect theory in economics: A review and assessment. *Journal of Economic Perspectives*, *27*(1), 173–196.

Bimbaum, M. H. (1982). Controversies in psychological measurement. In B. Wegener (Ed.), *Social attitudes and psychophysical measurement* (pp. 401–485). NJ: Erlbaum.

Birnbaum, M. H. (2018). Empirical evaluation of third-generation prospect theory. *Theory and Decision*, *84*(1), 11–27.

Birnbaum, M. H., Coffey, G., Mellers, B. A., & Weiss, R. (1992). Utility measurement: Configural-weight theory and the judge's point of view.. *Journal of Experimental Psychology: Human Perception and Performance*, *18*(2), 331.

Birnbaum, M. H., & Stegner, S. E. (1979). Source credibility in social judgment: Bias, expertise, and the judge's point of view.. *Journal of Personality and Social Psychology*, *37*(1), 48.

Birnbaum, M. H., & Sutton, S. E. (1992). Scale convergence and utility measurement. *Organizational Behavior and Human Decision Processes*, *52*(2), 183–215.

Birnbaum, M. H., Yeary, S., Luce, R. D., & Zhao, L. (2016). Empirical evaluation of four models of buying and selling prices of gambles. *Journal of Mathematical Psychology*, *75*, 183–193.

Birnbaum, M. H., & Zimmermann, J. M. (1998). Buying and selling prices of investments: Configural weight model of interactions predicts violations of joint independence. *Organizational Behavior and Human Decision Processes*, *74*(2), 145–187.

Carmona, R. (2008). *Indifference pricing: theory and applications*. Princeton University Press.

Chudziak, J. (2020). On complementary symmetry under cumulative prospect theory. *Journal of Mathematical Psychology*, *95*, Article 102312.

Dubourg, W. R., Jones-Lee, M. W., & Loomes, G. (1994). Imprecise preferences and the WTP-WTA disparity. *Journal of Risk and Uncertainty*, *9*(2), 115–133.

Eisenberger, R., & Weber, M. (1995). Willingness-to-pay and willingness-to-accept for risky and ambiguous lotteries. *Journal of Risk and Uncertainty*, *10*(3), 223–233.

Giraud, R. (2010). On the interpretation of the WTP/WTA gap as imprecise utility: an axiomatic analysis. *Economics Bulletin*, *30*(1), 692–701.

Hershey, J. C., & Schoemaker, P. J. (1985). Probability versus certainty equivalence methods in utility measurement: Are they equivalent? *Management Science*, *31*(10), 1213–1231.

Hey, J. D., Morone, A., & Schmidt, U. (2009). Noise and bias in eliciting preferences. *Journal of Risk and Uncertainty*, *39*(3), 213–235.

Irwin, J. R. (1994). Buying/selling price preference reversals: Preference for environmental changes in buying versus selling modes. *Organizational Behavior and Human Decision Processes*, *60*(3), 431–457.

Kahneman, D., & Tversky, A. (1979). Prospect theory: an analysis of decision under risk. *Econometrica*, *47*(2), 263–292.

Kontek, K., & Lewandowski, M. (2018). Range-dependent utility. *Management Science*, *64*(6), 2812–2832.

Kőszegi, B., & Rabin, M. (2006). A model of reference-dependent preferences. *Quarterly Journal of Economics*, *121*(4), 1133–1165.

Kőszegi, B., & Rabin, M. (2007). Reference-dependent risk attitudes. *American Economic Review*, *97*(4), 1047–1073.

Lewandowski, M. (2018). Complementary symmetry in cumulative prospect theory with random reference. *Journal of Mathematical Psychology*, *82*, 52–55.

Luce, R. D. (1991). Rank-and sign-dependent linear utility models for binary gambles. *Journal of Economic Theory*, *53*(1), 75–100.

Luce, R. D. (2000). *Utility of gains and losses: measurement-theoretical and experimental approaches*. Lawrence Erlbaum Associates Publishers.

Markowitz, H. (1952). The utility of wealth. *Journal of Political Economy*, *60*(2), 151–158.

McClelland, G. H., & Schulze, W. D. (1991). The disparity between willingness-to-pay versus willingness-to-accept as a framing effect. In *Frontiers of mathematical psychology* (pp. 166–192). Springer.

Mellers, B. A., Ordoñez, L. D., & Birnbaum, M. H. (1992). A change-of-process theory for contextual effects and preference reversals in risky decision making. *Organizational Behavior and Human Decision Processes*, *52*(3), 331–369.

Savage, L. J. (1954). *The foundations of statistics*. Wiley.

Sayman, S., & Öncüler, A. (2005). Effects of study design characteristics on the WTA–WTP disparity: A meta analytical framework. *Journal of Economic Psychology*, *26*(2), 289–312.

Schmidt, U., Starmer, C., & Sugden, R. (2008). Third-generation prospect theory. *Journal of Risk and Uncertainty*, *36*(3), 203–223.

Schmidt, U., & Traub, S. (2009). An experimental investigation of the disparity between WTA and WTP for lotteries. *Theory and Decision*, *66*(3), 229–262.
Schneider, M., & Day, R. (2018). Target-adjusted utility functions and expected-utility paradoxes. *Management Science*, *64*(1), 271–287.
Sugden, R. (2003). Reference-dependent subjective expected utility. *Journal of Economic Theory*, *111*(2), 172–191.
Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, *5*(4), 297–323.
Viscusi, W. K., & Huber, J. (2012). Reference-dependent valuations of risk: Why willingness-to-accept exceeds willingness-to-pay. *Journal of Risk and Uncertainty*, *44*(1), 19–44.
Wakker, P. P. (2020). A one-line proof for complementary symmetry. *Journal of Mathematical Psychology*, *98*, Article 102406.