# Super-replicating portfolios

1. Introduction

Assume that in one year from now the price for a stock X may take values in the set $\{\$1, \$2, \ldots, \$100\}$.

Consider four derivative instruments and their payoffs which depends on the stock price in one year:

a) One stock X
   - payoff $a_i^1 = i$
b) Zero-coupon bond with maturity date in one year
   - payoff $a_i^2 = 1$
c) European put option with expiration date in one year, strike price is $40
   - payoff $a_i^3 = \max(40 - i, 0)$
d) European call option with expiration date in one year, strike price is $60
   - payoff $a_i^4 = \max(i - 60, 0)$
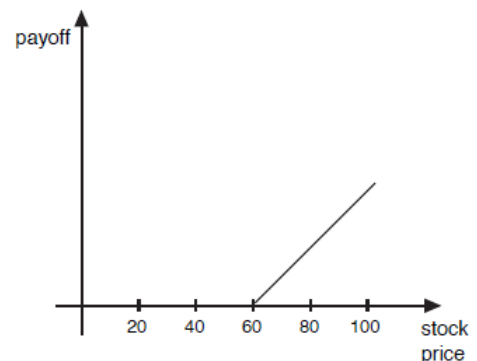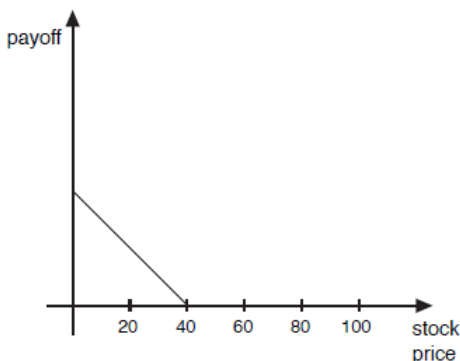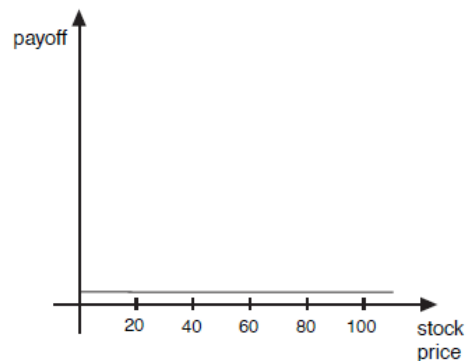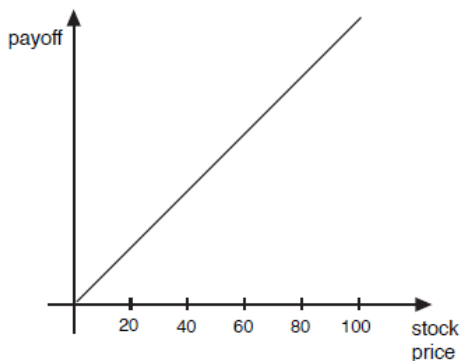
1.1.   In general

Payoff matrix: $P = [a^1 \ a^2 \ \dots \ a^N]$

Assume there are N instruments and M possible stock prices (states of the world in general)

Investment banks offer the so called **structured products** for investors with special needs.

1.2.   Example (hedging currency risk)

A company is planning to set up marketing operations in a foreign country. It is motivated by favorable sales projections. However, the company faces significant risks that are unrelated to its products or operation. If the Dollar value of the foreign currency depreciates, it may operate at a loss even if sales projections are met. In addition, if the economy of the country is unstable, there may be a dramatic devaluation arising from poor economic conditions. In this event, the sales projections would become infeasible and the company would pull out of the country entirely.

The anticipated profit over the coming year from operating in this foreign country is

- p Dollars, so long as the Dollar value of the currency remains at its current level r_0,
- $p\frac{r_1}{r_0}$ Dollars, if the Dollar value were to change to r_1,
- 0 Dollars, if the exchange rate drops to a critical value of $r_1 = r^*$.

The company consults with an investment bank, expressing a desire to meet its profit projection during its first year of operation by focusing on product sales, without having to face risks associated with the country's currency and economic conditions. The bank designs a structured product that offers a payoff in one year that is contingent on the prevailing exchange rate r_1. The payoff function is illustrated in the Figure below. By purchasing this structured product, the company can rest assured that its profit over the coming year from this foreign operation will be p Dollars, so long as its sales projections are met. When selling a structured product, an investment bank may be taking on significant risks.
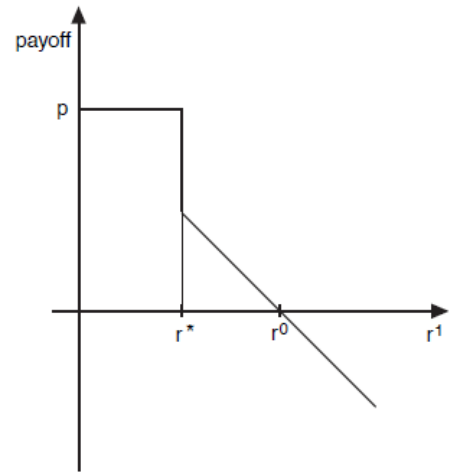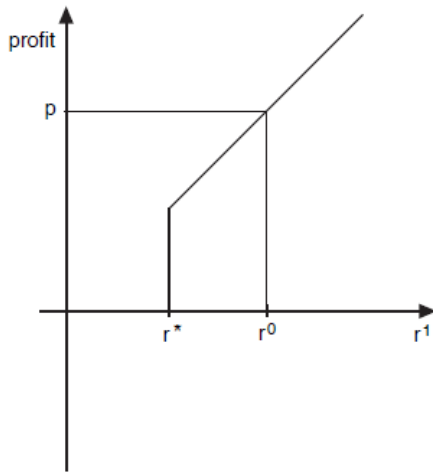
2. Replicating the structured product

By selling such instrument, the bank takes all the currency and political risk from the investor. This risk may be avoided by replicating the structured product with the instruments available on the market. We say that the structured product with payoff vector $b \in R^M$ may be **replicated** with a portfolio of assets available on the market if

$$Px = b,$$

for some $x \in R^N$, which denotes the portfolio. Negative values represent „**short selling**" of a given product. Short selling occurs when we borrow the asset from a broker, sell it and give it back to the broker in the future, after we buy the same amount of asset back from the market.

If $Px = b$, payoff of the protfolio and of the structured product are identical, so by buying this portfolio the bank may avoid the whole risk (perfect hedging) and acts therefore as a mediator between the market and the client.

Can any structured product be replicated with assets available on the market?

Instrument with payoff $b \in R^M$ may be replicated if and only if $Px = b$ for some $x \in R^N$.

So in order to replicate each instrument, $Px = b$ has to have a solution for each $b \in R^M$. It is true if $\text{rank}(P) = M$. Then we say that the **market is complete**.

3. Pricing and arbitrage

Until now we concetrated on payoffs. And what if we introduce prices?

Let $\rho \in R^N$ be the price vector of assets available in the market. A portfolio $x \in R^N$ requires $\rho^T x$ amount of investment.

The bank should price the structured product as the price for replicating portfolio (+ the margin). Otherwise there will be arbitrage.

Arbitrage occurs if there exists a portfolio $x \in R^N$, such that from a negative investement $\rho^T x < 0$ we can get nonnegative profit $Px \geq 0$.

3.1. Example (Put-Call parity)

Consider 4 assets:

a) stock, current payoff s_0, price in one month $s_1 \in \{1, 2, \ldots, 100\}$;
   - payoff a^1;
b) zero-coupon bond, current price β_0, maturity date in one month;
   - payoff a^2;
c) European put option, current price p_0 with a strike price $\kappa > 0$, expiration date in one month;
   - payoff a^3;
d) European call option, current price c_0 with a strike price $\kappa > 0$, expiration date in one month;
   - payoff a^4;

If we buy one stock, one put option and short sell one call option and κ units of a zero-coupon bond, our payoff will be zero:

$$a^1 - \kappa a^2 + a^3 - a^4 = i - \kappa + \max(\kappa - i, 0) - \max(i - \kappa, 0) = 0$$

The price of such portfolio must be zero in order to avoid arbitrage:

$$s_0 - \kappa \beta_0 + p_0 - c_0 = 0$$

This is the so called **put-call parity**.

## 4. Super-replicating the structured product

If the market is not complete (if $\text{rank}(P) < M$), then not every instrument may be replicated.

Then we want to find the **cheapest super-replicating portfolio**, (super-replicating means $Px \geq b$)

### 4.1. Example (currency risk continued)

We continue the same example:

- $r^0 = 0.5$, $r^* = 0.1$, $p = \$10 \ mln$, $r^1 \in \{0.01, 0.02; \dots, 0.99, 1.00\}$
- The structured product payoff vector belongs to R^{100}:
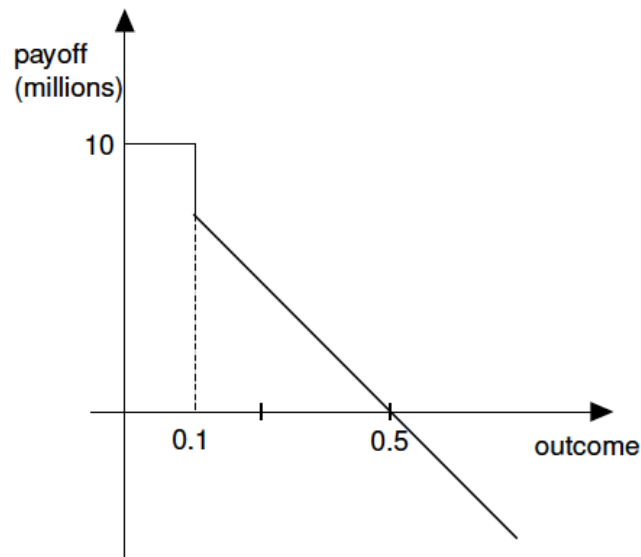
We have the following instruments available on the market:

a) Currency, payoff in one year $a^1 \in R^{100}, a^1 = i/100$, for each i;
b) Zero-coupon bond, payoff in one year $a^2 \in R^{100}, a^2 = 1$, dla każdego i;
c) European call option, strike price 0.1; payoff in one year $a^3 \in R^{100}, a^3 = \max(a_i^1 - 0.1; 0)$, for each i;
d) European call option, strike price 0.2; payoff in one year $a^4 \in R^{100}, a^4 = \max(a_i^1 - 0.2; 0)$, for each i;
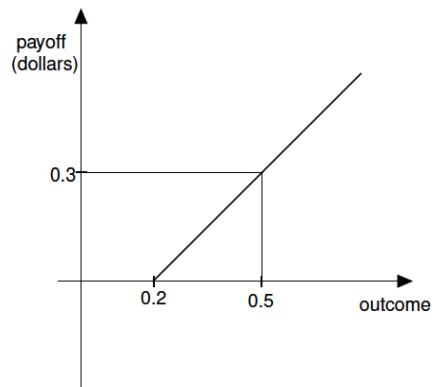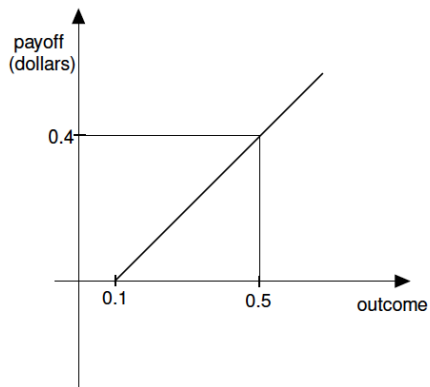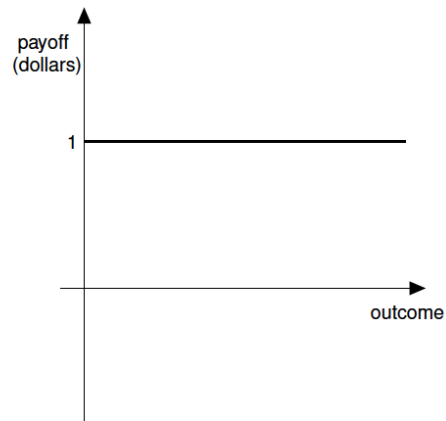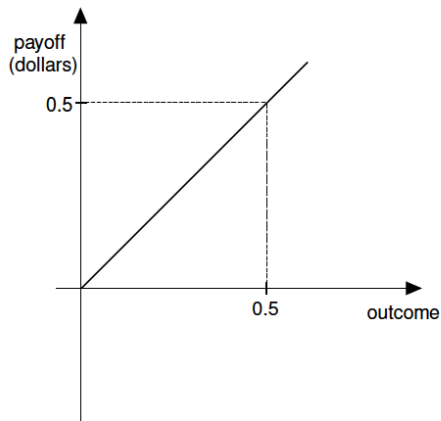
Payoff vector of the structured product does not belong to the space spanned by payoffs of the four available assets. Hence, the product may not be replicated exactly. It should be super-replicated. For example one could buy 10 mln zero-coupon bonds – the payoff would then be 10 mln Dollars irrespective of the exchange rate. We are interested in the cheapest super-replicating portfolio. The alternative which is better than 10 mln zero-coupon bonds is the following:

- Buy 10 mln zero-coupon bonds
- Buy 20 mln call options with strike price 0.2
- Short sell 40 mln call options with strike price 0.1
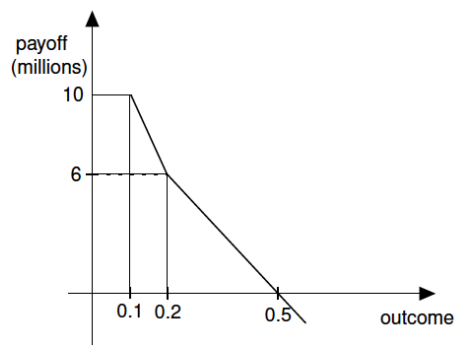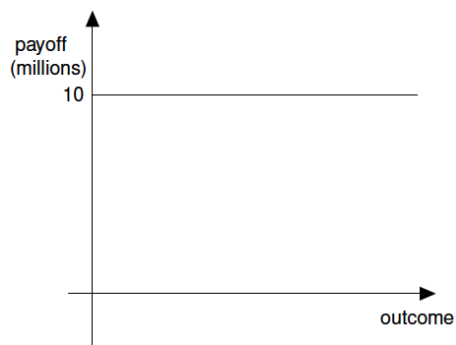
The mathematical program which makes it possible to find the cheapest super-replicating portfolio is as follows:

$$\min \rho^T x$$
$$s.t. \quad Px \geq b$$

Suppose that the proces for one unit of currency, zero-coupon Bond and both types of call options are, respectively: $0.5, $0.9, $0.4, $0.35. Then the cheapest super-replicating portfolio is indeed the one proposed above, i.e. $x = 10^6 \times [0 \; 10 \; -40 \; 20]^T$.

In the optimum at east N constraints must be binding. Hence, Px^* is equal to the b vector at least for N values of the exchange rate.

5. Detecting arbitrage

We want to find a portfolio x such that $\rho^T x < 0$ and $Px \geq 0$. By buying assets in this portfolio we get $-\rho^T x > 0$ and we don't have to pay for it later (free lunch). But why stop there? Why not buying $100x$? Arbitrage let's us earn as much as we want.

Mathematical program to find arbitrage:

$$\min \rho^T x$$
$$s.t. \; Px \geq 0$$
$$\rho^T x = -1$$

This program finds arbitrage portfolio that brings us income of 1 (+inf will never stop iterating).

Mathematical program to find arbitrage portfolio with minimal number of transactions:

$$\min e^T(x^+ + x^-)$$
$$p.w. \quad P(x^+ - x^-) \geq 0$$
$$\rho^T(x^+ - x^-) = -1$$
$$x^+ \geq 0$$
$$x^- \geq 0$$

Similarly, we construct a problem, when we want to minimize transaction costs – we would put transaction costs vector instead of e in the program above.