Decision theory - lecture notes

Michał Lewandowski

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1 Expected Utility motivation

1. Expected Value principle: problem of points or problem of division of the stakes posed by Chevalier de Mere to Blaise Pascal https://en.wikipedia.org/wiki/Problem_of_points; Pascal exchaged letters with Pierre de Fermat and found a solution which was the basis of Expected Value principle. Then Christiaan Huygens (1657) published the first treatise on probability. Explicitly the term Expected Value was first used by Laplace (1814). https://en.wikipedia.org/wiki/Expected_value. Since then it was widely believed that EV is a good measure for the attractiveness of a gamble. Two examples:
    - Chevalier de Mere problem: Should you bet 1:1 on at least one six occuring in four independent throws of a dice? Or should you bet 1:1 on at least one pair of sixes occuring in twenty four throws of a dice?
    - Chuck-a-luck: one throw of three dices; three sixes: you win 300; two sixes: you win 200; one six: you win 100; no sixes: you lose 100. Should you accept?
2. Saint Petersburg paradox: Nicolas Bernoulli (1713) in a letter to Pierre de Montmort suggested the following lottery: you throw a fair coin many times. Your payoff is 2^k, $k \in \mathbb{N}$, if tail occurs on k-th throw for the first time. https://en.wikipedia.org/wiki/St._Petersburg_paradox What would be a fair price to pay the casino for entering the game? Expected Value is infinite: $EV = \frac{1}{2}\times 2+\frac{1}{4}\times 4+\ldots+\frac{1}{2^k}\times 2^k+\ldots = 1+1+\ldots = \infty$. Yet people rarely want to pay more than 10 to play this game. Daniel Bernoulli (1738) proposed a solution: "The determination of the value of an item must not be based on the price, but rather on the utility it yields". Instead of expected value of the lottery prize, he suggested to use expected value of the utility of wealth and suggested that this utility is logarithmic: $u(w) = \log(w)$ where $w > 0$ is the decision maker's wealth.
3. Relative returns: We can arrive at the same conclusion by considering relative returns instead of nominal income. Suppose first that the decision maker invests the amount S and wants his long-term relative return to be at least 1. Define a multiplicative lottery, in which the return equals $2^k/S$, with probability $\frac{1}{2^k}$, for $k \in \mathbb{N}$, if tail occurs on k-th

throw for the first time. The average relative return is calculated by a geometric mean of a multiplicative lottery. The idea is that if there is equal chance of doubling your wealth (return of 2) or otherwise loosing half of it (return of $\frac{1}{2}$), the average relative return is $\left(\frac{1}{2} \times 2\right)^{0.5} = 1$ (using geometric mean) instead of $\frac{1}{2} \times 2 + \frac{1}{2} \times \frac{1}{2} = \frac{5}{4}$ (using simple arithmetic mean). In our case we use a generalized geometric mean with (possibly) unequal weights/probabilities. We obtain the following:

$$\prod_{i=1}^{\infty} \left(\frac{2^i}{S}\right)^{1/2^i} \geqslant 1.$$

After taking the logarithm on both sides we get:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \log(2^i) \geqslant \log(S).$$

Note that it is the expected logarithm of the lottery payoff as suggested by Daniel Bernoulli. Now It is a matter of some algebraic manipulations to show that $S \leqslant 4$. Suppose that we use the logarithm base 2 for convenience:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \log(2^i) = \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{8} \times 3 + \ldots$$

To calculate this sum we shall use a neat trick. Note that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots = 1$. On the other hand $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots = \frac{1}{2}$, and $\frac{1}{8} + \frac{1}{16} \ldots = \frac{1}{4}$. Continuing this way, we can see that summing over all these sums we will get the sum we are looking for, and since summing the left hand side of these sums is the same as summing the right hand sides of these sums we get:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \log(2^i) = \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{8} \times 3 + \ldots$$
$$= 1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2$$

Finally we get $2 \geqslant \log(S)$ or $S \leqslant 4$, as claimed. Hence we see that the amount invested should not be more than 4 if we want to ensure average return of at least 1.

4. Relative returns and buying/selling price: Denote a lottery by $\mathbf{x}$. Mathematically it is equivalent to finite support random variable. In general we need initial wealth W (a budget) to cover possible losses resulting from playing a lottery. Suppose that preferences are logarithmic.

By the argument made above logarithmic preferences correspond to the decision maker who cares about relative returns and calculates average return or geometric mean. Selling price of lottery $\mathbf{x}$ is the minimal price the agent is willing to accept to forego the right to play this lottery, i.e. a minimal $S \in \mathbf{R}$, for which $W + S \succsim W + \mathbf{x}$. Such S is determined by solving the following equation:

$$\log(W + S) = \mathbf{E}\log(W + \mathbf{x}).$$

Note that the calculation we made in point 3 above is a special case of selling price S, when $W = 0$. On the other hand, buying price of lottery $\mathbf{x}$ is the maximal price the agent is willing to pay for the right to play this lottery, i.e. a maximal $B \in \mathbf{R}$, for which $W + \mathbf{x} - B \succsim W$. Such B is determined by solving the following equation:

$$\mathbf{E}\log(W + \mathbf{x} - B) = \log(W).$$

5. As an example consider lottery $(+120, \frac{1}{2}; -100, \frac{1}{2})$ (equal chance of winning 120 or losing 100) and $W = 1000$. Selling price is calculated as:

$$\frac{1}{2}\log(900) + \frac{1}{2}\log(1120) = \log(1000 + S)$$
$$(900 \times 1120)^{0.5} - 1000 = S$$

So $S = 3.992$. Similarly buying price is calculated as:

$$\frac{1}{2}\log(900 - B) + \frac{1}{2}\log(1120 - B) = \log(1000)$$
$$\log\left[(900 - B)(1120 - B)\right]^{0.5} = \log(1000)$$
$$900 \times 1120 - B(900 + 1120) + B^2 = 1000^2$$

So $B = 3.968$.

6. Relative returns and riskiness measure: Suppose that preferences are logarithmic so that we are interested in relative returns. Let $\mathbf{x}$ be a real-valued random variable with finite support such that it has positive expected value ($\mathbf{E}\mathbf{x} > 0$) and negative consequences in the support ($\mathbb{P}[\mathbf{x} < 0] > 0$). Call such RV lottery. For such lottery there is a unique W^* such that $W^* + \mathbf{x} \sim W^*$. (you are asked to prove it). It is determined by solving the following equation:

$$\mathbf{E}\log(W^* + \mathbf{x}) = \log(W^*).$$

Such W^* is called the riskiness measure of $\mathbf{x}$ and will be denoted by $R(\mathbf{x})$. For more on the riskiness measure see the article `http://www.ma.huji.ac.il/hart/abs/risk.html?`

7. As an example consider lottery $(+120, \frac{1}{2}; -100, \frac{1}{2})$. This lottery satisfies the conditions mentioned above $\mathbf{E}\mathbf{x} = 10 > 0$ and $\mathrm{P}[\mathbf{x} < 0] = \frac{1}{2} > 0$. Observe that if $W = 200$, then $\frac{1}{2}\log(100) + \frac{1}{2}\log(320) < \log(200)$ because $\log\left[\frac{100}{200} \times \frac{320}{200}\right]^{0.5} = \log\left[\frac{32}{40}\right]^{0.5} < 0$. On the other hand if $W = 1000$, then $\frac{1}{2}\log(900) + \frac{1}{2}\log(1120) < \log(1000)$ because $\log[0.9 \times 1.12]^{0.5} = \log[1.008]^{0.5} > 0$. Finally it is easy to verify that $W^* = 600$, because $\log\left[\frac{500}{600} \times \frac{720}{600}\right]^{0.5} = \log\left[\frac{5}{6} \times \frac{6}{5}\right]^{0.5} = 0$.

8. Define buying price and selling price for a lottery $\mathbf{x}$ as functions of wealth and denote it by $B(W, \mathbf{x})$ and $S(W, \mathbf{x})$, respectively. There are simple relationships between riskiness measure R and buying and selling prices:

$$\begin{aligned} S(W, \mathbf{x} - B(W, \mathbf{x})) &= 0 \\ S(W - B(W, \mathbf{x}), \mathbf{x}) &= B(W, \mathbf{x}) \\ B(W + S(W, \mathbf{x}), \mathbf{x}) &= S(W, \mathbf{x}) \\ R(\mathbf{x} - B(W, \mathbf{x})) &= W \\ R(\mathbf{x} - S(W, \mathbf{x})) &= W + S(W, \mathbf{x}) \end{aligned}$$

Make sure that you understand why the above relationships hold.

2 Expected Utility Theory

In this section we follow Kreps Microeconomic Theory textbook.

1. Let X be the set of prizes. A simple probability distribution $P : X \to [0,1]$ is specified by:

   a) a finite subset of X, called the support of P and denoted by supp(P), and

   b) for each $x \in \text{supp}(P)$ a number $P(x) > 0$, with $\sum_{x\in\text{supp}(P)} P(x) = 1$.

   The set of simple probability distributions on X will be denoted by $L(X)$.

2. Degenerate lottery: the lottery that gives the prize x with probability one will be written by δ_x.

3. Mixing operation: given $P, Q \in L(X)$ and $\alpha \in [0,1]$, we define $\alpha P + (1-\alpha)Q \in L(X)$, such that $(\alpha P + (1-\alpha)Q(x) = \alpha P(x) + (1-\alpha)Q(x)$, for $x \in X$.

4. Let $\succ \subseteq L(X) \times L(X)$ be a strict preference relation. Let $\succsim, \sim \subseteq L(X) \times L(X)$ be defined as: $P \succsim Q$, if $Q \not\succ P$, $P \sim Q$, if $P \not\succ Q \wedge Q \not\succ P$. We impose the following axioms on $\succ$:

Axiom 1 (Weak order). *$\succ$ is asymmetric ($P \succ Q \Rightarrow Q \not\succ P$) and negatively transitive ($P \not\succ Q \wedge Q \not\succ R \Rightarrow P \not\succ R$)*

Remarks:

i) Reminder: $\succ$ is asymmetric if and only if $\succsim$ is complete, and $\succ$ is negatively transitive if and only if $\succsim$ is transitive. (Very simple proofs are left for the reader.)

ii) $P \sim Q$ means that the DM is equally happy with P as she is with Q. It does not mean that she is unable to judge. Completeness requires that she is always able to express her preferences. Violations of completeness are analyzed by Aumann (1962) `http://www.jstor.org/stable/1909888` and Dubra, et al. (2004) `http://www.sciencedirect.com/science/article/pii/S0022053103001662`

iii) violations of transitivity are usually excluded based on the money pump argument (but it requires continuity as well): suppose that $A \succ$

$B \succ C \succ A$. Then I propose the following deal: take A, exchange A for C and pay me ϵ, exchange C for B and pay me ϵ, exchange B for A and pay me ϵ. By continuity I can always find $\epsilon > 0$ small enough such that the DM is willing to do that. But with each cycle she looses $3\epsilon > 0$ for sure.

iv) Weak order (plus a technical axiom of Separability – see Cantor, 1915) alone gives us the following representation: $\succ$ satisfies Axiom 1 and Separability if and only if there exists a function $U : L(X) \to \mathbf{R}$, such that

$$P \succ Q \iff U(P) > U(Q), \quad \text{for any } P, Q \in L(X).$$

Moreover, this function is unique up to strictly increasing transformation.

Axiom 2 (Substitution/Independence). *Let $P, Q \in L(X)$ such that $P \succ Q$. Let $\alpha \in (0, 1)$, and $R \in L(X)$. Then $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R$.*

Axiom 3 (Archimedean/Continuity). *Let $P, Q, R \in L(X)$ such that $P \succ Q \succ R$. Then there exist numbers $\alpha, \beta \in (0, 1)$, such that $\alpha P + (1-\alpha)R \succ Q \succ \beta P + (1-\beta)R$.*

Theorem 1. *A preference relation $\succ \subseteq L(X) \times L(X)$ satisfies Axioms 1-3 if and only if there is a function $u : X \to \mathbf{R}$ such that:*

$$P \succ Q \iff \sum_{x \in \text{supp}(P)} u(x)P(x) > \sum_{x \in \text{supp}(Q)} u(x)Q(x).$$

Moreover, if u provides a representation of $\succ$ in this sense, then v does as well if and only if there exist $a, b \in \mathbf{R}$, $a > 0$, such that $v(\cdot) \equiv au(\cdot) + b$.

Proof: We assume that there are $b, w \in X$, such that $b \succ P \succ w$, for any $P \in L(X)$. Furthermore we assume that $b \succ w$.

Lemma 1. *For any numbers $\alpha, \beta \in [0, 1]$, $\alpha\delta_b + (1-\alpha)\delta_w \succ \beta\delta_b + (1-\beta)\delta_w$ if and only if $\alpha > \beta$.*

Proof. We prove the following intermediate result:

$$P \succ Q, \ \alpha \in (0, 1) \Rightarrow P \succ \alpha P + (1-\alpha)Q \succ Q. \tag{1}$$

To prove it, apply Independence twice. Now we prove the Lemma. If one only of $\alpha = 1$, $\beta = 0$ holds, then Lemma 1 reduces to (1). If both hold, the conclusion coincides with the assumption. Assume that $0 < \beta < \alpha < 1$. Then by (1), $P \succ \alpha P + (1-\alpha)Q \succ Q$. Note that $\beta P + (1-\beta)Q = \gamma[\alpha P + (1-\alpha)Q] + (1-\gamma)Q$, with $0 < \gamma = \beta/\alpha < 1$. So by (1), $\alpha P + (1-\alpha)Q \succ \beta P + (1-\beta)Q$. To prove the converse implication, we use the proven implication and completeness of $\succsim$. □

Lemma 2. *For any degenerate lottery δ_x, there is $\alpha \in [0, 1]$, such that $\delta_x \sim \alpha\delta_b + (1-\alpha)\delta_w$.*

Proof. Take any $x \in X$ and consider two sets:

$$\{\alpha \in [0, 1] : \ \alpha\delta_b + (1-\alpha)\delta_w \succ \delta_x]\}$$
$$\{\alpha \in [0, 1] : \ \alpha\delta_b + (1-\alpha)\delta_w \prec \delta_x]\}$$

These sets are nonempty (see assumption), open (by continuity A3) and disjoint by asymmetry of $\succ$. Hence they cannot cover the whole interval $[0, 1]$.There must be a number $\alpha_x \in [0, 1]$, such that $\alpha_x\delta_b + (1-\alpha_x)\delta_w \sim \delta_x$. (Then Lemma 1 implies uniqueness of α_x.) □

Lemma 3. *If $P \sim Q$, and $R \in L(X)$, $\alpha \in [0, 1]$, then $\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)R$.*

Proof. We prove the following intermediate result:

$$P \sim Q, \ \alpha \in (0, 1) \Rightarrow P \sim \alpha P + (1-\alpha)Q \sim Q. \tag{2}$$

By contradiction suppose that $P \sim Q$, but $P \succ \alpha P + (1-\alpha)Q =: S$. Let $T := 1/2P + 1/2S$. By (1), $P \succ T \succ S$. On the other hand, $Q \sim P \succ T$, and S is between Q and T, so for some $\delta \in (0, 1)$, $S = \delta Q + (1-\delta)T$. Since $Q \succ T$, by (1) we get $S \succ T$, a contradiction.

Now we prove the Lemma. If $R \sim P \sim Q$, then the Lemma coincides with (2). Assume that $R \succ P \sim Q$, and $\alpha \in (0, 1)$. Define $S := \alpha P + (1-\alpha)R$ and $T := \alpha Q + (1-\alpha)R$. By contradiction suppose that $S \succ T$. Since $R \succ P$, by (1), $R \succ S$. So $R \succ S \succ T$. By continuity (A3), there is $\beta \in (0, 1)$, such that $S \succ V$, where $V := \beta R + (1-\beta)T$. Note that $V = \beta R + (1-\beta)T = \beta R + (1-\beta)(\alpha Q + (1-\alpha)R) = (\alpha - \alpha\beta)Q + (1-\alpha+\alpha\beta)R$.

Let $W := \beta R + (1-\beta)Q$. By (1), $R \succ W \succ Q \sim P$. By independence (A2), $\alpha W + (1-\alpha)R \succ \alpha P + (1-\alpha)R = S$. Observe that $\alpha W + (1-\alpha)R = \alpha(\beta R + (1-\beta)Q) + (1-\alpha)R = (\alpha - \alpha\beta)Q + (1-\alpha+\alpha\beta)R = V$. So $V \succ S$, a contradiction. □

The rest if easy: For every prize x, define $u(x) \in [0, 1]$, such that:

$$\delta_x \sim u(x)\delta_b + (1-u(x))\delta_w.$$

This number $u(x)$ will be the utility of the prize x. We know that this number exists by Lemma 2 and is unique by Lemma 1. Take any lottery P.

Lemma 4. *For $u : X \to \mathbf{R}$ defined as above, for any lottery P, the following holds:*

$$\left(\sum_{x\in\text{supp}(P)} u(x)P(x)\right)\delta_b + \left(1 - \sum_{x\in\text{supp}(P)} u(x)P(x)\right)\delta_w.$$

Proof. Take any $P \in L(X)$, with n outcomes, denoted by x_i and probabilities denoted by p_i. It can be written as: $\sum p_i\delta_{x_i}$, or for some j, $p_j\delta_{x_j} + (1 - p_j)\sum_{i\neq j}\frac{p_i}{1-p_j}\delta_{x_i}$. By Lemma 2 and 1 there is a unique number $u(x_j)$, such that $\delta_{x_j} \sim u(x)\delta_b + (1 - u(x))\delta_w$. By Lemma 3,

$$p_j\delta_{x_j} + (1 - p_j)\sum_{i\neq j}\tfrac{p_i}{1-p_j}\delta_{x_i} \sim p_j u(x)\delta_b + p_j(1 - u(x))\delta_w + (1 - p_j)\sum_{i\neq j}\tfrac{p_i}{1-p_j}\delta_{x_i}$$

Having replaced δ_{x_j} by a mixture of δ_w and δ_b, we can now do the same for all x_i, $i \neq j$. Each time we will get an equivalent lottery with a different δ_{x_i}, replaced by a mixture of δ_w and δ_b. By transitivity of indifference and Lemma 3 each such new lottery is equivalent to P. After n steps, we obtain $(\sum_i p_i u(x_i))\delta_b + (1 - \sum_i p_i u(x_i))\delta_w$, what was to be proved. □

We now prove the uniqueness part of the Theorem. Suppose that U and V are two affine functions that represent $\succ \subseteq L(X) \times L(X)$. Take a lottery $P \in L(X)$. Since U represents preferences, $U(P) = \alpha_P U(b) + (1 - \alpha_P)U(w)$, which implies that $\alpha_P = \frac{U(P)-U(w)}{U(b)-U(w)}$, and $P \sim \alpha_P\delta_b + (1 - \alpha_P)\delta_w$. Since V represents the same preferences

$$\begin{aligned}
V(P) &= V(\alpha_P\delta_b + (1 - \alpha_P)\delta_w) \\
&= \alpha_P V(b) + (1 - \alpha_P)V(w) \\
&= \frac{U(P) - U(w)}{U(b) - U(w)}V(b) + \frac{U(b) - U(P)}{U(b) - U(w)}V(w) \\
&= \frac{V(b) - V(w)}{U(b) - U(w)}U(P) + V(w) - U(w)\frac{V(b) - V(w)}{U(b) - U(w)}
\end{aligned}$$

Let $c = \frac{V(b)-V(w)}{U(b)-U(w)}$, and $d = V(w) - U(w)\frac{V(b)-V(w)}{U(b)-U(w)}$. We have proved that there exist constants c, d, $c > 0$, such that $V(\cdot) = cU(\cdot) + d$.

3 EU axioms and result - discussion

1. TODO: Machina triangle and independence
2. TODO: Discussion on other axioms

4 Risk attitudes within Expected Utility

1. Let X be a subset of $\mathbf{R}$. Instead of the probability distribution $P \in L(X)$ we will now alternatively say lottery for a real-valued random variable $\mathbf{x}$, whose associated distribution is P. Let $\mathcal{X}$ be the set of all such random variables. A degenerate random variable with one element support $x^* \in X$ will be denoted simply by x^*.

2. An individual whose preference relation is $\succ \subseteq \mathcal{X} \times \mathcal{X}$ is risk averse if $\mathbf{E}[\mathbf{x}] \succ \mathbf{x}$, for every lottery $\mathbf{x}$. Suppose that the preference relation satisfied the vNM axioms. It means that it can be represented by a vNM utility function u, and the condition for risk aversion can be written as: $u(\mathbf{E}[\mathbf{x}]) < \mathbf{E}u(\mathbf{x})$, for every non-degenerate $\mathbf{x}$. This is Jensen's inequality, which is true if and only if u is strictly concave. Similarly, the decision maker who always prefers a gamble to its expected value is risk-loving: $\mathbf{E}[\mathbf{x}] \prec \mathbf{x}$, which leads to a convex utility function, and the DM who is indifferent between a gamble and its expected value is risk neutral: $\mathbf{E}[\mathbf{x}] \sim \mathbf{x}$, which leads to a linear utility.

3. For a random variable $\mathbf{x}$ define its Certainy Equivalent $\text{CE}(\mathbf{x}) \in X$, such that $\text{CE}(\mathbf{x}) \sim \mathbf{x}$. Using the EU representation, we have $u(\text{CE}(\mathbf{x})) = \mathbf{E}u(\mathbf{x})$. Recalling what we have established in point 2. we can now say: the decision maker is risk-averse/risk neutral/risk-loving if $\text{CE}(\mathbf{x}) \lesseqgtr \mathbf{E}(\mathbf{x})$, for every non-degenerate gamble $\mathbf{x}$.

4. Let $\mathbf{x}, \mathbf{y}$ be two lotteries with the associated CDFs F_x and F_y, respectively. We say that $\mathbf{x}$ First Order Stochastically Dominates $\mathbf{y}$, written $\mathbf{x} \succ_{\text{FOSD}} \mathbf{y}$, if $F_x(t) \leqslant F_y(t)$, for all $t \in \mathbf{R}$, and $F_x \neq F_y$. In this case $\mathbf{x}$ can be constructed from $\mathbf{y}$ be a series of upward mass shifts.

5. Let P_x, P_y be probability functions of $\mathbf{x}$ and $\mathbf{y}$, respectively. First Order Stochastic Dominance is a partial order. Expected Utility is consistent with FOSD if $\mathbf{x} \succ_{\text{FOSD}} \mathbf{y}$ implies $P_x \succ P_y$, for all $\mathbf{x}, \mathbf{y}$ (or alternatively it implies $\mathbf{E}u(\mathbf{x}) > \mathbf{E}u(\mathbf{y})$).

   Theorem 2. *Expected Utility Theorem is consistent with FOSD if and only if the vNM utility function is strictly increasing.*

   Proof. Let us suppose that u is continuously differentiable. We want to prove that $F_x(t) \leqslant F_y(t)$, for all $t \in \mathbf{R}$, $F_x \neq F_y$ implies $\mathbf{E}u(\mathbf{x}) - \mathbf{E}u(\mathbf{y}) >$

0, if and only if u is strictly increasing. Let us integrate by parts:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} u(t)(dF_x(t) - dF_y(t)) \\
&= \left[u(t)(F_x(t) - F_y(t))\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} u'(t)(F_y(t) - F_x(t))dt \\
&= \int_{-\infty}^{+\infty} u'(t)(F_y(t) - F_x(t))dt
\end{aligned}$$

If u is strictly increasing, $u'(t) > 0$, for all $t \in \mathbf{R}$. It implies that the sum above must be strictly positive by FOSD. Conversely, if the sum is strictly positive, then u' must be strictly positive for all $t \in \mathbf{R}$ by FOSD. □

6. Let $\mathbf{x}, \mathbf{y}$ be two lotteries with the associated CDFs F_x and F_y, respectively. We say that $\mathbf{x}$ Second Order Stochastically Dominates $\mathbf{y}$, written $\mathbf{x} \succ_{\text{SOSD}} \mathbf{y}$, if $\int_{-\infty}^{t} F_x(s)ds \leqslant \int_{-\infty}^{t} F_y(s)ds$, for all $t \in \mathbf{R}$, and $G_x \neq G_y$, where $G_x(t) = \int_{-\infty}^{t} F_x(s)ds$ and $G_y(t) = \int_{-\infty}^{t} F_y(s)ds$. In this case $\mathbf{x}$ can be constructed from $\mathbf{y}$ be a series of Mean Preserving Spreads, i.e. $\mathbf{y} = \mathbf{x} + \epsilon$, where ϵ is a nondegenerate random variable with mean 0, interpreted as: $\mathbf{y}$ is the same as $\mathbf{x}$, except for $\mathbf{y}$ contains additional pure risk ϵ.

7. Let P_x, P_y be probability functions of $\mathbf{x}$ and $\mathbf{y}$, respectively. Second Order Stochastic Dominance is a partial order. Expected Utility is consistent with SOSD if $\mathbf{x} \succ_{\text{SOSD}} \mathbf{y}$ implies $P_x \succ P_y$, for all $\mathbf{x}, \mathbf{y}$ (or alternatively it implies $\mathbf{E}u(\mathbf{x}) > \mathbf{E}u(\mathbf{y})$).

   Theorem 3. *Assume that u is strictly increasing. Expected Utility Theorem is consistent with SOSD if and only if the vNM utility function is strictly concave.*

   Proof. Let us suppose that u is twice continuously differentiable. We want to prove that $\int_{-\infty}^{t} F_x(s)ds \leqslant \int_{-\infty}^{t} F_y(s)ds$, for all $t \in \mathbf{R}$, and $G_x \neq G_y$, where G_x, G_y are defined above, implies $\mathbf{E}u(\mathbf{x}) - \mathbf{E}u(\mathbf{y}) > 0$, if and only if u is strictly increasing and strictly concave. We already know the first intergretion. Let us integrate it by parts once more:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} u(t)(dF_x(t) - dF_y(t)) \\
&= \int_{-\infty}^{+\infty} u'(t)(F_y(t) - F_x(t))dt \\
&= \left[u'(t) \int_{-\infty}^{t} (F_y(s) - F_x(s))ds \right]_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} u''(t) \int_{-\infty}^{t} (F_y(s) - F_x(s))dsdt
\end{aligned}$$

If u is strictly increasing and strictly concave ($u' > 0$, $u'' < 0$), then by SOSD, the above difference must be strictly positive. Conversely, if the above difference is strictly positive, then u'' must be strictly negative by SOSD. □

8. Let u be the vNM utility of an agent. Risk aversion is measured locally by Arrow (1965), Pratt (1964) absolute risk aversion coefficient $ARA(x) = -\frac{u''(x)}{u(x)}$.

   - It could be measured by the curvature of the utility function $-u''$. However, the utility function is unique up to strictly increasing affine transformation. So if u and v both represent the same preferences, $v(\cdot) = au(\cdot) + b$, where $a, b \in \mathbf{R}$, $a > 0$. So $-u''(\cdot)$ and $-v''(\cdot) = -au''(\cdot)$ differ from each other by a, and they should not. Hence we normalize the risk aversion measure by the first derivative. Then $-\frac{u''(x)}{u''(x)} = -\frac{v''(x)}{v'(x)} = -\frac{au''(x)}{au'(x)}$, for all $x \in \mathbf{R}$.

9. Let us characterize the decision maker who exhibits constant absolute risk aversion, i.e. $ARA(x) = -\frac{u''(x)}{u(x)} = \alpha$, for all $x \in \mathbf{R}$, where $\alpha \in \mathbf{R}$ is a constant.

   Proposition 1. *The decision maker exhibits Constant Absolute Risk Aversion (CARA) if and only if her preferences are represented by a utility function $u_\alpha : \mathbf{R} \to \mathbf{R}$, for some $\alpha \in \mathbf{R}$, normalized w.l.o.g. by $u(0) = 0$ and $u'(0) = 1$ and defined as:*

$$u_\alpha(x) = \begin{cases} \frac{1-e^{-\alpha x}}{\alpha}, & \alpha \neq 0, \\ x, & \alpha = 0. \end{cases}$$

   Proof. We solve the differential equation $ARA(x) = -\frac{u''(x)}{u(x)} = \alpha$ satisfying the two initial conditions $u(0) = 0$ and $u'(0) = 1$. This is a Cauchy problem. The equation can be written as follows:

$$\begin{aligned}[\log u'(x)]' &= -\alpha \,|\int dx\\ \log u'(x) &= -\alpha x + c\\ u'(x) &= \exp(-\alpha x + c)\end{aligned}$$

where c represents the constant of integration. Using the first initial condition $u'(0) = \exp(c) = 1$, so $c = 0$. Integrating the equation once again we obtain: $u(x) = -\frac{\exp(-\alpha x)}{\alpha} + d$, where d represents another constant of intergration. Using the second initial condition $u(0) = -\frac{1}{\alpha} + d = 0$, so $d = \frac{1}{\alpha}$ and we get $u(x) = \frac{1-\exp(-\alpha x)}{\alpha}$, for $\alpha \neq 0$. Finally using the d'Hospital rule:

$$\lim_{\alpha\to 0} \frac{1-\exp(-\alpha x)}{\alpha} \overset{\text{H}}{=} \lim_{\alpha\to 0} \frac{x\exp(-\alpha x)}{1} = x \qquad \square$$

10. Let u be a vNM utility function. There is also another way to measure risk aversion. This is via relative risk aversion coefficient $RRA(x) = -\frac{u''(x)x}{u'(x)}$, for all $x \in [0, \infty)$. Let us characterize the decision maker who exhibits constant relative risk aversion, i.e. $RRA(x) = -\frac{u''(x)x}{u(x)} = \beta$, for all $x \in [0, \infty)$, where $\beta \in \mathbf{R}$ is a constant.

Proposition 2. *The decision maker exhibits Constant Relative Risk Aversion (CRRA) if and only if her preferences are represented by a utility function $u_\beta : [0, \infty) \to \mathbf{R}$, for some $\beta \in \mathbf{R}$, normalized w.l.o.g. by $u(1) = 0$ and $u'(1) = 1$ and defined as:*

$$u_\beta(x) = \begin{cases} \frac{x^{1-\beta}-1}{1-\beta}, & \beta \neq 1,\\ \log(x), & \beta = 1.\end{cases}$$

Proof. We solve the differential equation $RRA(x) = -\frac{u''(x)x}{u(x)} = \beta$ satisfying the two initial conditions $u(1) = 0$ and $u'(1) = 1$. This is a Cauchy problem. The equation can be written as follows:

$$\begin{aligned}[\log u'(x)]' &= -\beta[\log x]' \,|\int dx\\ \log u'(x) &= -\beta \log x + c\\ u'(x) &= \exp(c)x^{-\beta}\end{aligned}$$

where c represents the constant of integration. Using the first initial condition $u'(1) = \exp(c) = 1$, so $c = 0$. Integrating the equation once again we obtain: $u(x) = \frac{1}{1-\beta}x^{1-\beta} + d$, where d represents another constant of intergration. Using the second initial condition $u(1) = \frac{1}{1-\beta} + d = 0$, so $d = -\frac{1}{1-\beta}$ and we get $u(x) = \frac{x^{1-\beta}-1}{1-\beta}$, for $\beta \neq 1$. Finally using the d'Hospital rule:

$$\lim_{\beta\to 1} \frac{x^{1-\beta}-1}{1-\beta} \overset{H}{=} \lim_{\beta\to 1} \frac{-x^{1-\beta}\log x}{-1} = \log x \qquad \square$$

11. Observe that the CRRA class belongs to decreasing absolute risk aversion class (DARA), because if $RRA(x) = \beta = \text{const}$, then $ARA(x) = \frac{\beta}{x}$, which is decreasing in x.

12. TODO: Wealth and scale invariant strategies and the corresponding Vincze functional equations.

13. TODO: Pratt's theorem of comparative risk aversion

5 Expected Utility example applications

1. Demand for insurance example: a strictly risk-averse DM has initial wealth W. She can lose the amount D of her wealth with probability π. She can buy insurance that costs q per unit and pays 1 per unit conditional on the loss occurring. She must decide how many units of insurance, α, to buy. The lottery she faces can be written as: $l := (W - \alpha q, 1 - \pi; W - \alpha q - D + \alpha, \pi)$. The expected value of this lottery equals:

   $$E(l) = W - \alpha q - \pi(D - \alpha).$$

   Let us state the proposition:

   Proposition 3. *If insurance is actuarially safe, a strictly risk averse DM will choose to fully insure against risk.*

   Proof. Suppose that the DM's preferences are represented by utility function u. Her problem might be written as:

   $$\max_{\alpha} \mathbf{E}u(l) = (1 - \pi)u(W - \alpha q) + \pi u(W + (1 - q)\alpha - D).$$

   Assuming interior solution $\alpha^* > 0$, it has to satisfy the FOC for this optimization problem:

   $$-q(1 - \pi)u'(W - \alpha^* q) + \pi(1 - q)u'(W + (1 - q)\alpha^* - D) = 0.$$

   Suppose that the price of insurance is actuarially fair. It means that the insurance company has zero profits from insurance (neither gains nor loses); her profits are $\alpha q - \pi\alpha$ and they are zero if $q = \pi$. Let's assume that this is the case. Then FOC becomes:

   $$u'(W + (1 - q)\alpha^* - D) = u'(W - \alpha^* q).$$

   Since u' is strictly decreasing (the DM is risk averse), we have:

   $$W + (1 - q)\alpha^* - D = W - \alpha^* q \;\Rightarrow\; \alpha^* = D. \qquad \square$$

2. Demand for risky assets example: there are two assets; a safe asset with a return of 1 and a risky asset with a random return z, distributed according to F on the support $[a, b] \subseteq \mathbf{R}$, such that $E(\mathbf{z}) > 1$. The decision maker has an initial amount of wealth W, invests an amount α in the risky asset, $W - \alpha$ in the safe asset. Her preferences are represented by a utility function u. It is assumed that she is risk averse for all levels of wealth.

Proposition 4. *If a risky asset is actuarially favorable, then any risk averse individual will always buy some of it.*

Proof. The decision maker's wealth after buying a portfolio equals $W + (\mathbf{z} - 1)\alpha$. Her utility maximization problem can be written as:

$$\max_{\alpha \in [0,W]} \int_a^b u(W + (z-1)\alpha) dF(z).$$

Let α^* be an argument that maximizes the above expression. It has to satisfy the following Kuhn-Tucker first-order condition:

$$\int_a^b (z-1)u'(W + (z-1)\alpha^*) dF(z) \begin{cases} < 0 & \text{if } \alpha^* = 0, \\ = 0 & \text{if } \alpha^* \in (0, W), \\ > 0 & \text{if } \alpha^* = W. \end{cases} \tag{3}$$

The second order condition is:

$$\int_a^b (z-1)^2 u''(W + (z-1)\alpha^*) dF(z).$$

It is satisfied because $(z-1)^2 \geqslant 0$, for any $z \in [a, b]$ and the second derivative of u is always negative due to risk aversion. We will prove the proposition by contradiction. Suppose that $\alpha^* = 0$. Note that the FOC evaluated at $\alpha^* = 0$ becomes $\int_a^b (z-1)u'(W) dF(z) = u'(W)(E(\mathbf{z}) - 1)$, which is positive because $E(\mathbf{z}) > 1$. Hence $\alpha^* = 0$ does not satisfy the FOC. $\square$

6 Problems with Expected Utility

1. TODO: the Allais type paradoxes: common ratio, common consequence effects.

2. TODO: the list of other EU paradoxes: WTA/WTP disparity, preference reversal, coexistence of insurance and gambling, framing effects, etc.

3. EU is locally risk neutral: Barberis, Huang, Thaler (2006) made an experiment involving three groups of people: MBA students, financial analysts and very rich investors, and asked them whether they would accept an equal chance of either winning \$550 or losing \$500? 71% turned down the gamble. Suppose that $\mathbf{x}$ is a gamble with an equal chance of either winning $(1+\lambda)x$ dollars or losing x dollars. Let u be a vNM utility function and W an initial wealth. The EU of the gamble is $Eu(W+\mathbf{x}) = \frac{1}{2}u(W-x) + \frac{1}{2}u(W+(1+\lambda)x)$. Let us see what is the effect of increasing x at $x^* = 0$.

$$\begin{aligned} \left.\frac{dEu(W+\mathbf{x})}{dx}\right|_{x=0} &= -\tfrac{1}{2}u'(W) + \tfrac{1}{2}(1+\lambda)u'(W) \\ &= \tfrac{1}{2}\lambda u'(W) \end{aligned}$$

   It means that if λ is positive (the gamble is actuarially favorable), then the DM will accept the gamble for small x. This means that Expected Utility is locally risk neutral (Arrow, 1971). So in order to explain that people reject the gamble for small x we need to introduce loss aversion.

   Definition 1. *We say that the DM exhibits loss aversion if his preferences are represented by a utility function u, such that $u(x) < -u(-x)$, for all x.*

4. Suppose that the utility function takes the following form:

$$u(x) = \begin{cases} \bar{u}(x), & \text{for } x \geqslant 0, \\ -\lambda\bar{u}(-x), & \text{for } x < 0. \end{cases}.$$

   where $\lambda > 0$ and $\bar{u}$ is a strictly increasing utility function with $\bar{u}(0) = 0$. When λ is greater than 1 the decision maker exhibits loss aversion.

5. Rabin (2000) paradox: modest risk aversion over small stakes implies unrealistically high risk aversion over large stakes:

Proposition 5. *If a EU maximizer rejects* $P := (\$110, \frac{1}{2}; -\$100, \frac{1}{2})$ *at any initial wealth level, then she will reject* $Q := \left(\infty, \frac{1}{2}; -\$1000, \frac{1}{2}\right)$ *at any wealth level.*

This is called a paradox since while the premise sounds reasonable, the conclusion does not. The idea is that the utility function u of a EU maximizer with initial wealth W that rejects P must be very concave at W. Since this is true for all W, then the whole function u must be very concave, which means that it becomes flat very quickly.

Proof. A EU maximizer rejects lottery P at any wealth level W. It implies that the utility function must be concave and satisfies:

$$12u(W+110) + \tfrac{1}{2}u(W-100) \leqslant u(W)$$
$$\text{or equivalently:} \quad u(W+110) - u(W) \leqslant u(W) - u(W-100)$$

Note that a linear approximation of a concave function lies always on or above the function. Using this property we have:

$$\begin{aligned} u(W+110) &\geqslant u(W) + 110u'(W+110) \\ u(W) &\geqslant u(W-100) + 100u'(W-100) \end{aligned}$$

Putting it al together we obtain:

$$\begin{aligned} 110u'(W+110) &\leqslant u(W+110) - u(W) \\ &\leqslant u(W) - u(W-100) \\ &\leqslant 100u'(W-100) \\ \text{or equivalently:} \quad \frac{u'(W+110)}{u'(W-100)} &\leqslant \frac{100}{110}. \end{aligned}$$

If we do the same at wealth level $W + 210$ instead of W, we get:

$$\frac{u'(W+320)}{u'(W+110)} \leqslant \frac{100}{110}$$

Combining both inequalities:

$$\frac{u'(W+110)}{u'(W-100)} \frac{u'(W+320)}{u'(W+110)} \leqslant \left(\frac{100}{110}\right)^2$$

Continuing this way we get:

$$\frac{u'(W-100+210n)}{u'(W-100)} \leqslant \left(\frac{100}{110}\right)^n$$

For example if $n = 50$, then $210 \times 50 = 10500$ and $\left(\frac{100}{110}\right)^{50} = 0.008$. □

7 Prospect Theory

1. TODO: Original Prospect Theory
2. TODO: Problems with original Prospect Theory
3. TODO: Cumulative Prospect Theory
4. TODO: Some applications
5. TODO: Third-generation prospect theory

8 EU theory vs. EU models

1. TODO: Consequentialism and the EU of lifetime wealth model
2. TODO: Mental accounting, gambling wealth and the EU of gambling wealth model
3. TODO: Reference dependence and the Reference-dependent EU model

9 Range-dependent utility

1. TODO: Range-dependent utility as a general model and decision utility as its operational special case