PREFERENCE RELATION

Modified and adapted from materials written by Michal Jakubczyk

- Preferences:
  - capability of making comparisons
  - capability of deciding, which of two alternatives is better/is not worse (we’re not talking about making a choice)

- Mathematically – binary relations in the set of decision alternatives:
  - **X** – decision alternatives
  - $\mathbf{X^2}$ – all pairs of decision alternatives
  - $\mathbf{R \subset X^2}$ – binary relation in **X**, selected subset of ordered pairs of elements of **X**, say first element is preferred
  - if **x** is in relation **R** with **y**, then we write **xRy** or $\mathbf{(x,y) \in R}$

Binary relations – example #1

- Example
  - X={1,2,3,4}
  - R – a relation denoting „is smaller than”
  - xRy – means „x is smaller than y”

- Thus:
  - (1,2)∈R; (1,3)∈R; (1,4)∈R; (2,3)∈R; (2,4)∈R; (3,4)∈R
  - 1R2, 1R3, 1R4, 2R3, 2R4, 3R4
  - eg. (2,1) doesn’t belong to R

	1	2	3	4
1		√	√	√
2			√	√
3				√
4				

Binary relations– example #2

- Example
  - X={1,2,3,4}
  - R – a relation with no (easy) interpretation (?)
  - R={(1,2), (1,3), (2,3), (2,4), (3,2), (4,4)}

	1	2	3	4
1		√	√	
2			√	√
3		√		
4				√

Binary relations – basic properties

- complete: **xRy** or **yRx**
- reflexive: **xRx**
- irreflexive: not **xRx**
- transitive: if **xRy** and **yRz**, then **xRz**
- symmetric: if **xRy**, then **yRx**
- asymmetric: if **xRy**, then not **yRx**
- antisymmetric: if **xRy** and **yRx**, then **x=y**
- negatively transitive: if not **xRz** and not **zRy**, then not **xRy**
  - or alternatively: if **xRy**, then either **xRz** or **zRy**
- acyclic: if $x_1Rx_2R\ldots Rx_n$, then $x_1 \neq x_n$

Exercise – check the properties of the following relations

- R_1: (among people), to have the same colour of the eyes
- R_2: (among people), to know each other
- R_3: (in the family), to be an ancestor of
- R_4: (among real numbers), not to have the same value
- R_5: (among words in English), to be a synonym
- R_6: (among countries), to be at least as good in a rank-table of summer olympics

	R_1	R_2	R_3	R_4	R_5	R_6
complete						
reflexive						
irreflexife						
transitive						
symmetric						
asymmetric						
antisymmetric						
negatively transitive						

Preference relation

- Preferences – capability of making comparisons, of selecting not worse an alternative out of a pair of alternatives
  - we'll talke about selecting a strictly better (or just as good) alternative later on

- Depending on its preferences we'll use one of the relations:
  - Preorder
  - Partial order
  - Weak order (rational preferences)
  - Linear order

Preorder

- R is **a preorder** in X, if it is:
  - reflexive
  - transitive

- We do not want R to be:
  - complete – we cannot compare all pairs
  - antisymmetric – if **xRy** and **yRX**, then not necessarily **x=y**

Preorder – an example

- Michał is at a party and can pick from a buffet onto his plate: small tartares, cocktail tomatoes, sushi (maki), chunks of cheese
- A decision alternative is an **orderd four-tuple**, denoting number of respective goodies, there can be **at most 20 pcs** on the plate
- Michał **preferes more** pcs than fewer. **At the same time**, he prefers **more tartere** than less. Michał **cannot tell**, if he wants to have more pcs if it mean less tartare.

Element	Formal notation
Set of alternatives X	
Relation R „at least as good as”	
Is R reflexive? … transitive? … total? … antisymmetric?	

Partial order

- R is **a partial order** in X, if it is:
  - reflexive
  - transitive
  - antisymmetric (not needed in the preorder)

- We do not want it to be:
  - complete – we cannot compare pairs

Partial order – an example

- Michał is at a party …
- Decision alternatives are **ordered pairs** : # of pcs, # of tartares
- Conclusion – different structure (of the same problem), different formal representation

Element	Formal notation
Set of alternatives X	
Relation R „at least as good as”	
Is R reflexive? … transitive? … total? … antisymmetric?	

Preference relation

- R is a **weak order** in X, if it is:
  - complete
  - transitive
- Completeness implies reflexivity
- We do not want it to be:
  - antisymmetric – equally good alternatives can differ
- In our example – if Michał didn't value tartare especially (and just wanted to eat as much as possible)

Linear order

- R is a **total order** in X, if it is:
  - complete (thus reflexive)
  - transitive
  - Antisymmetric

- In our example:
  - Michał wants to eat as much as possible
  - we represent alternatives as # of pcs

Preference relations

	Preorder	Partial order	Weak order	Linear order
reflexive	√	√	√	√
complete			√	√
transitive	√	√	√	√
antisymmetric		√		√

Preference and indifference relation

- Let R be a weak order (transitive, complete, thus reflexive)
  - **xRy** means „**x** is at least as good as **y**"

- R generates strict preference relation – P:
  - **xPy**, if **xRy** and not **yRx**
  - **xPy** means „**x** is better than **y**"

- R generates indifference relation – I:
  - **xIy**, if **xRy** and **yRx**
  - **xIy** means „**x** just as good as **y**"

An exercise

- X={a,b,c,d}
- R={(a,a), (a,b), (a,c), (a,d), (b,a), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d)}

- Find P and I

- P={(a,c), (a,d), (b,c), (b,d), (c,d)}
- I={(a,a), (a,b), (b,a), (b,b), (c,c), (d,d)}
- R=P∪I

Properties of P and I (of previous slides)

- Let P and I be generated by R – a weak order

- P is:
  - irreflexive
  - asymmetric
  - acycylic
  - transitive
  - negatively transitive

- I is an equivalence relation:
  - reflexive
  - transitive
  - symmetric

Proof of the properties of I ($xIy \Leftrightarrow xRy \wedge yRx$)

- reflexive (**xIx**)
  - obviuos – using reflexivity of R we get **xRx**
- transitive (**$xIy \wedge yIz \Rightarrow xIz$**)
  - predecessor equivalent to **$xRy \wedge yRx \wedge yRz \wedge zRy$**
  - using transitivity we get **$xRz \wedge zRx$**, QED
- symmetric (**$xIy \Rightarrow yIx$**)
  - predecessor equivalent to **$xRy \wedge yRx$**, QED

Proof of the properties of P (xPy $\Leftrightarrow$ xRy $\wedge$ ~ yRx)

- irreflexive (**~xPx**)
  - obvious – there cannot be **xRx** and not **xRx**

- asymmetric (**xPy** $\Rightarrow$ ~ **yPx**)
  - if **xPy**, then it meas that:
  - **xRy** $\wedge$ **~(yRx)**, and so it cannot be **yRx**, hence
  - **~(yPx)**

- transitive (**xPy** $\wedge$ **yPz** $\Rightarrow$ **xPz**)
  - we have **xRy** $\wedge$ **~(yRx)** $\wedge$ **yRz** $\wedge$ **~(zRy)**
  - then **xRz** (from transitivity of R), but can it be **zRx**?
  - (ad absurdum) assume **zRx**
  - then **zRy** – a contradiction, and so **~(zRx)**, thus **xPz**

- asyclic:
  - obvious from transitivity and antireflexivity

Proof of the properties of P ($xPy \Leftrightarrow xRy \wedge \sim yRx$)

- negatively transitive ($xPy \Rightarrow \forall z: xPz \vee zPy$)
  - we have **xPy**, and so $xRy \wedge \sim yRx$
  - take any **z** and assume that **~xPz** (if no then QED)
  - hence either **~xRz** or **zRx**
  - as R is total, then **zRx**
  - and so **zRy** (transitivity of R), can we have **yRz**?
  - no, cause it would mean **yRx** – a contradiction
  - hence $zRy \wedge \sim yRz$, and so **zPy**

Homework

1. Prove that

   $\mathbf{xRy} \wedge \mathbf{yPz} \Rightarrow \mathbf{xPz}$

2. Show that

   $\forall \mathbf{x,y}: \mathbf{xPy} \vee \mathbf{xIy} \vee \mathbf{yPx}$

Another definition of rational preferences

- Let's start with relation P:
  - asymmetric
  - negatively transitive

- Then we say that
  - **xIy**, if not ~**xPy** ∧ ~**yPx**
  - **xRy**, if **xPy** ∨ **xIy**

- Homework. Prove that with such definitions:
  - I is an equivalence relation
  - R is a weak order

Exercise

- $X=\{a,b,c,d\}$
- $P=\{(a,d), (c,d), (a,b), (c,b)\}$

- Find R and I

- $I=\{(a,a), (a,c), (b,b), (b,d), (c,a), (c,c), (d,b), (d,d)\}$
- $R=P\cup I$

Another definition of rational preferences

- Can wew start with I?
  - reflexive
  - symmetric
  - transitive

- No – we wouldn't be able to order the abstraction classes

Another definition of rational preferences

- Is it enough for P to require only:
  - asymmetric
  - acyclic (not necessarily negatively transitive)

- No – let's see an example

P from the previous slide – an example

- Mr X got ill and for years to come will have to take pills twice a day in an interval of exactly 12 hours. He can choose the time however.
- Mr X has very peculiar preferences – he prefers **y** to **x**, if $\mathbf{y=x+}\pi$, otherwise he doesn't care
- Thus **yPx**, if **y** lies on the circle π units farther (clockwise) than **x**

Exercise

- What properties does P have?
  - asymmetry
  - negative transitivity
  - transitivity
  - acyclicity

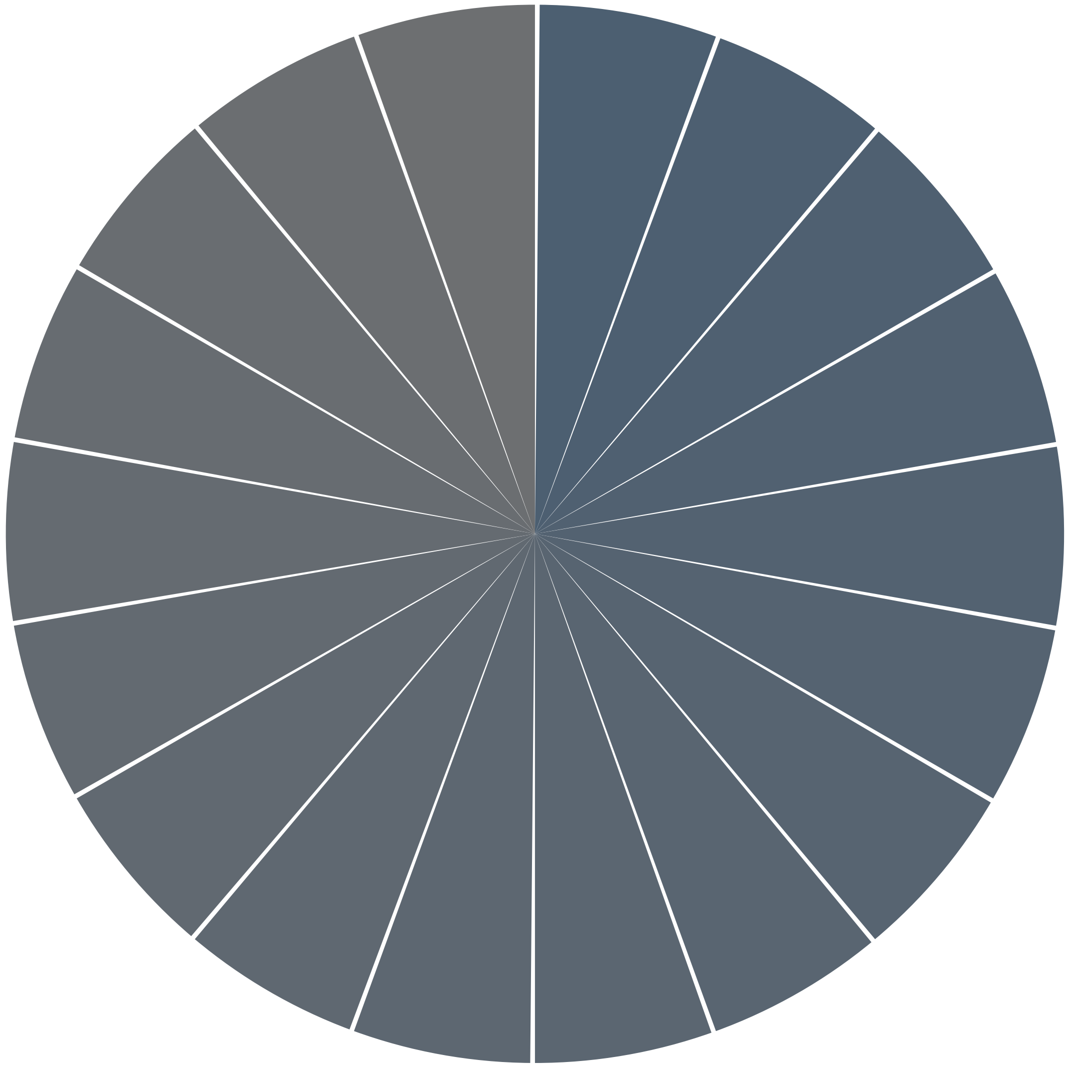
- P generates „weird” preferences:
  - $1+2\pi$ better than $1+\pi$,
    $1+\pi$ better than 1,
    $1+2\pi$ equally good as 1
  - 1 equally good as $1+\pi/2$,
    $1+\pi/2$ equally good as $1+\pi$,
    1 worse than $1+\pi$

Another definition of rational preferences

- What if we take P to be:
  - asymmetric
  - transitive (not necessarily negatively transitive)
  - thus acyclic

- First let's try to find an example
- Then let's think about such preferences

Asymmetric, transitive, not negatively transitive relation – intuition



Asymmetric, transitive, not negatively transitive relation – example

- **$X=\{R_+\}$, $xPy \Leftrightarrow x>y+5$** (I want more, but I am insensitive to small changes)

- Properties of **P**:
  - asymmetric – obviously
  - transitive – obviously
  - negatively transitive?
    - **11 P 5**, but
    - neither **11 P 8**, nor **8 P 5**

- Thus **I** is not transitive: **11 I 8** and **8 I 5**, but not **11 I 5**

- Real example – *non-inferiority testing*
  - H_0: $\mu_1=\mu_2$ vs H_1: $\mu_1\neq\mu_2$
  - H_0: $\mu_1\leq\mu_2-\delta$ vs H_1: $\mu_1>\mu_2$

Properties of preferences – a summary

P („better than") – asymmetric, negatively transitive

R („at least as good as") – transitive, complete

colours, insensitiviness to small changes

P („better than") – asymmetric, transitive

eg. Mr X

P („better than") – asymmetric, acyclic