Theory of decisions I - Take-home exam
Due date: June 21, noon

This is the final exam for Theory and decisions I. There will be no other assignments. Take your time and solve the problems at home.

If you have any questions regarding the problem formulation, please do not hesitate to ask. Remember, however, that all questions should be asked during office hours that will take place on **Thursday, June 13, room A-105**.

Good luck!!!

You are on vacations in Viganj, an excellent windsurfing spot in the Peljesac peninsula in Croatia. In the morning you need to decide whether to rent the windsurfing equipment (board and rig) or to take a ferry to the Korcula island and go to one of its nice beaches for the whole day. Your decision depends on the weather that day that is very uncertain in this period.

The relevant state space is the following:

$$S = \{ss, sw, cs, cw\},$$

where the first letter is either s for "sunny", or c for "cloudy", and the second letter is either s for "silent", or w for "windy". We assume that S is the universal event and the states in S are disjoint.

You only have partial information on the likelihood of different states that are relevant for your decision. Based on past weather statistics, you assess this likelihood as follows:

- The probability of "sunny" $\{ss, sw\}$ is no smaller than 0.3
- The probability of "cloudy" $\{cs, cw\}$is no smaller than 0.4
- The probability of "wind" $\{sw, cw\}$is no smaller than 0.5
- The probability of "silent" $\{ss, cs\}$ is no smaller than 0.2
- The probability of $\{ss, cs, cw\}$ is no smaller than 0.6

- The probability of $\{ss, sw, cw\}$ is no smaller than 0.9

These values are used to define a capacity. A capacity is a set function $v : 2^S \to [0,1]$ satisfying the following conditions:

1. $v(\emptyset) = 0$;
2. $A \subset B \ \Rightarrow \ v(A) \leq v(B), \quad \text{for } A, B \subset S$;
3. $v(S) = 1$.

We say that a capacity is convex if it satisfies

$$v(A) + v(B) \leq v(A \cap B) + v(A \cup B), \quad \text{for all } A, B \subset S.$$

Problem 1. *Based on the information you have to find such set function $v : 2^S \to [0,1]$ that:*

a) it provides the lowest bound of the probability for each subset of S, for example $v(\{cs, cw\}) = 0.3$,

b) it is a capacity,

c) this capacity is convex.

The set of possible consequences C contains the following elements:

$$\begin{aligned}
\text{funsurf} &= \text{surfing in the sun} \\
\text{oksurf} &= \text{surfing with some rain} \\
\text{nosurf} &= \text{waiting for the wind} \\
\text{funbeach} &= \text{fully enjoying the beach} \\
\text{okbeach} &= \text{enjoying the beach in the wind} \\
\text{nobeach} &= \text{no beach due to clouds and rain}
\end{aligned}$$

Your preferences towards sure consequences are as follows:

$$\text{funsurf} \sim \text{funbeach} \succ \text{okbeach} \succ \text{oksurf} \succ \text{nosurf} \sim \text{nobeach}$$

You have to decide between the two courses of actions:

States of nature	sw	cw	ss	cs
go surfing	funsurf	oksurf	nosurf	nosurf
go to beach	okbeach	nobeach	funbeach	nobeach

Your preferences over the AA acts satisfy the axioms of Schmeidler (1989) Choquet Expected Utility (i.e. the Anscombe-Aumann (1963) axioms where Independence is replaced with a weaker Comonotonic Independence). Suppose that there are two sources of uncertainty:

- subjective: the choice of states of nature
- objective: some probability experiments - see below

Having thought about your preferences for a while you determined the probability values p^*, p^{**} satisfying:

$$\begin{aligned}
\text{oksurf} &\sim (\text{funsurf}, p^*; \text{nosurf}, 1 - p^*) \\
\text{okbeach} &\sim (\text{funbeach}, p^{**}; \text{nobeach}, 1 - p^{**})
\end{aligned}$$

These values are: $p^* = 0.6$ and $p^{**} = 0.9$.

Problem 2. *Based on the information provided can you infer what are the utility values for all the consequences in C.*

Problem 3. *Calculate and compare the Choquet Expected Utility for the acts "go surfing" and "go to beach". Which one is preferred according to this model?*

Let v be the capacity over 2^S. The core of v (denoted by $\text{core}(v)$) is is the additive probability distribution $p \in [0,1]^{|S|}$ such that $p(s) \geq 0, \quad s \in S$, such that $\sum_{s\in S} p(s) = 1$ and $\forall A \subset S$: $p(A) \geq v(A)$.

Problem 4. *Find the core of a capacity v found in Problem 1. If it is not unique then it is a convex polygon. In this case, specify all of its vertices.*

Problem 5. *Now solve for* $\min_{p\in\text{core}(v)} \mathbf{E}u(\cdot)$ *for the act "go surfing" and "go to beach" [*$\mathbf{E}u(\cdot)$ *denotes the simple expected utility functional with respect to the respective probability measure]. You should use Excel Solver (or similar software) to do it - it is a simple linear programming problem.*

Problem 6. *Compare the values of the Choquet Expected Utility with the values found in Problem 5 for the acts "go surfing" and "go to beach".*

Problem 7. *How would the lottery* $(\text{funsurf}, 0.28; \text{nosurf}, 0.72)$ *be ranked against the acts "go surfing" and "go to beach"?*