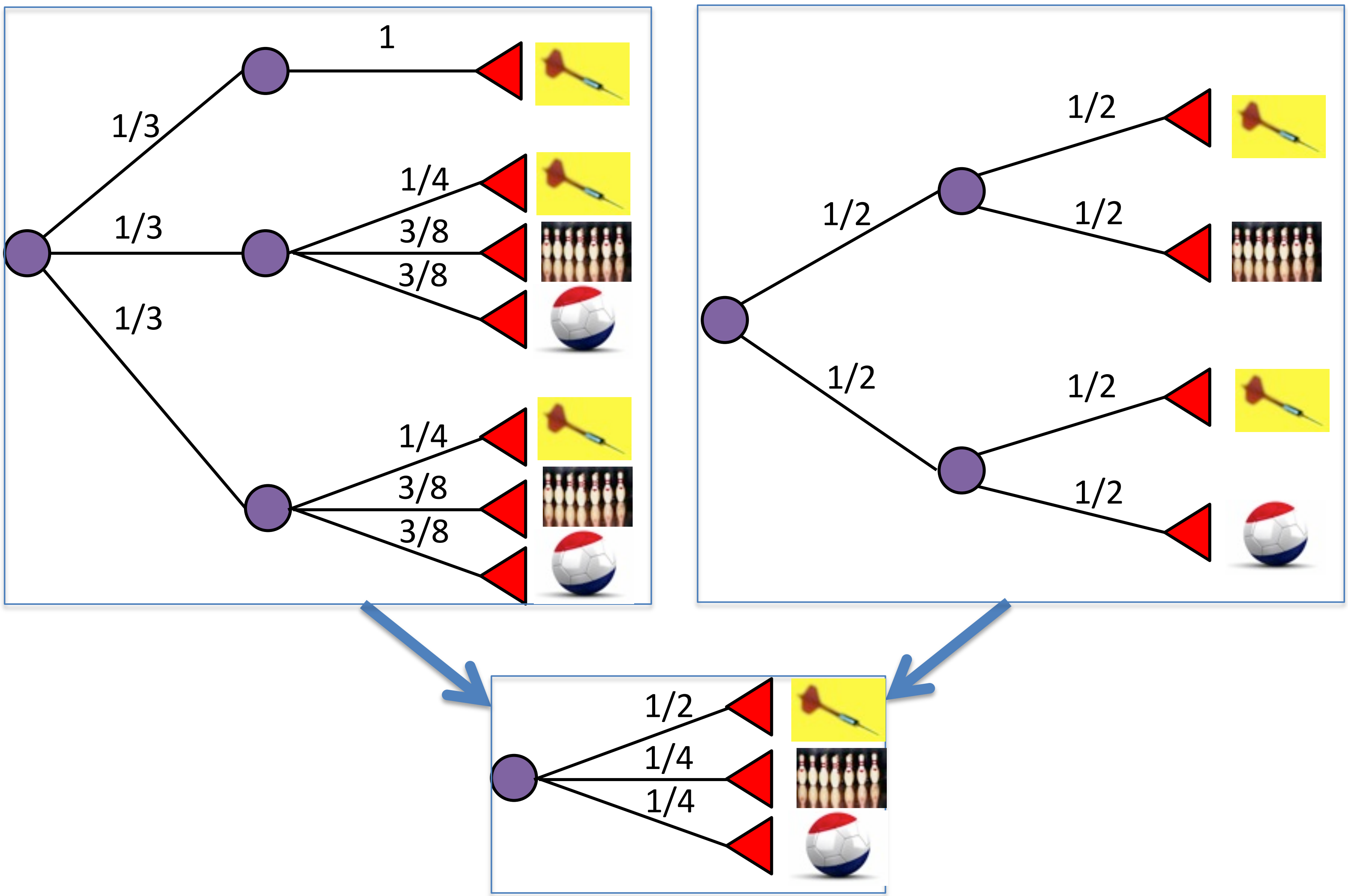
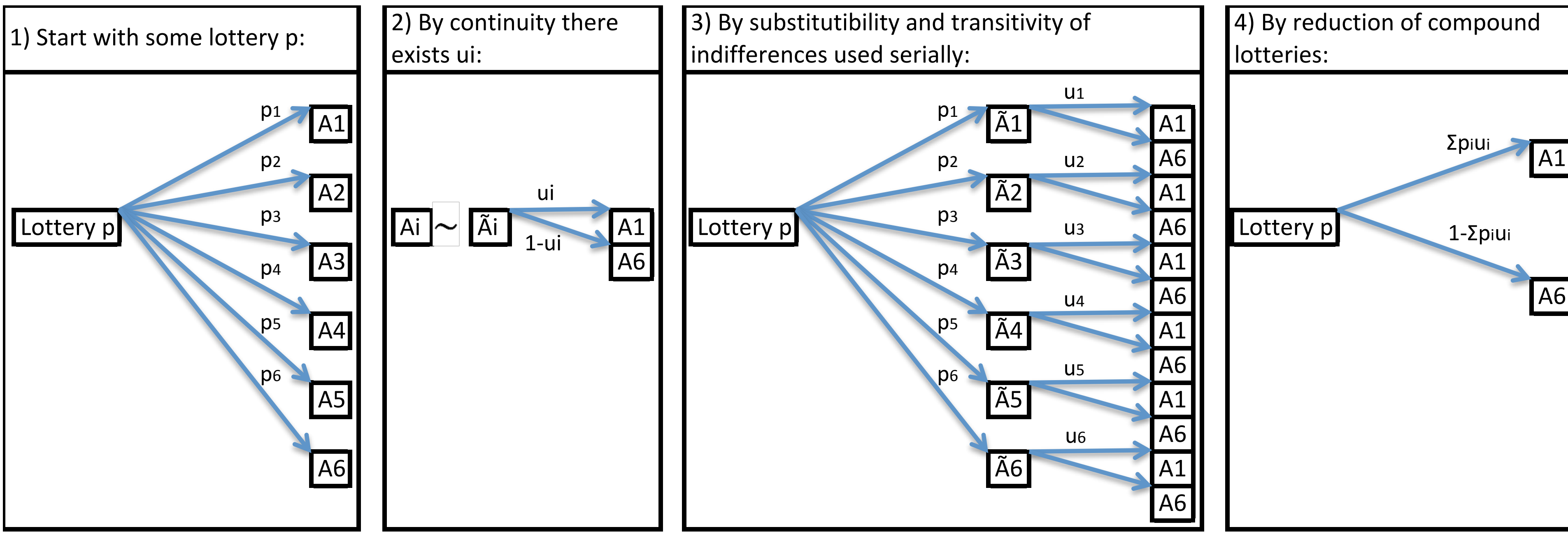
EU simple 2

Reduction of compound lotteries

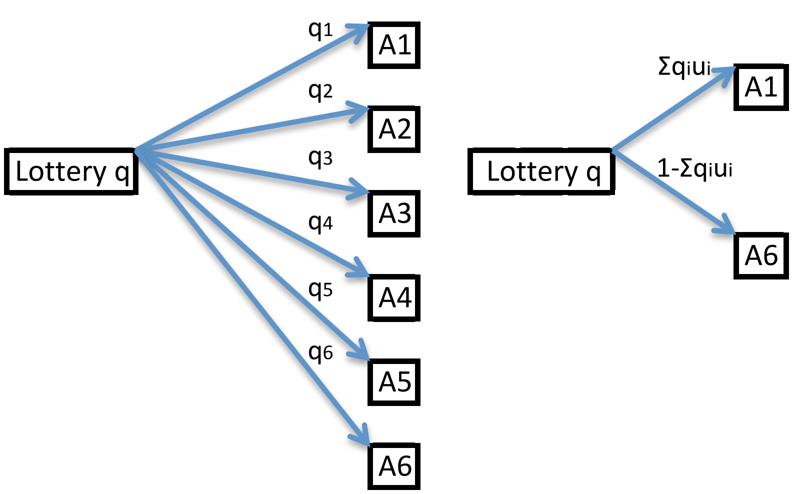


Von Neumann Morgenstern proof graphically 1



Von Neumann Morgenstern proof graphically 2

5) Repeat the whole procedure for another lottery q which you are comparing with lottery p



6) Finally, by monotonicity we have:

Lottery p weakly preferred to lottery q if and only if

$$\Sigma p_i u_i \geq \Sigma q_i u_i$$

Common misunderstandings

- The decision maker behaves as if he were a maximizer of expected values of utility

Some common falacies

Falacy (1)

Lottery $(A_1, p_1; \ldots; A_n, p_n)$ *is preferred to lottery* $(A_1, p'_1; \ldots; A_n, p'_n)$ *because the utility of the former,* $p_1 u_1 + \ldots + p_n u_n$*, is greater than the utility of the latter,* $p'_1 u_1 + \ldots + p'_n u_n$.

Some common falacies

Falacy (2)

Suppose that $A \succ B \succ C \succ D$ *and that the utilities of these alternatives satisfy* $u(A) + u(D) = u(B) + u(C)$*, then* $(B, \frac{1}{2}; C, \frac{1}{2})$ *should be preferred to* $(A, \frac{1}{2}; D, \frac{1}{2})$*, because although they have the same expected utility, but the former has smaller variance.*

Falacy (3)

Suppose that $A \succ B \succ C \succ D$ *and that the utilities of these alternatives satisfy* $u(A) - u(B) > u(C) - u(D)$*, then the change from* B *to* A *is more preferred than the change from* D *to* C

- In the ordinal utility representation for choice under certainty the crucial axiom "giving" order is TRANSITIVITY.
- Completeness is more a "technical" assumption needed in order to be able to make comparisons
- Here in the cardinal representation for choice under risk the crucial axiom "giving" cardinality is INDEPENDENCE
- Continuity is merely a technical assumption needed for existence of utility associations
- Monotonicity (which gives uniqueness of utility associations) and reduction of compound lotteries are not needed since they follow from other axioms in the general case
- On the next slide the more general formulation is presented

Axioms of von Neumann and Morgenstern

Axiom (Weak order)

$\succsim$ is complete and transitive

Axiom (Continuity)

For every $P, Q, R \in L$,

$$P \succ Q \succ R \Rightarrow \exists \alpha, \beta \in (0,1): \alpha P + (1-\alpha) R \succ Q \succ \beta P + (1-\beta) R$$

Axiom (Independence)

For every $P, Q, R \in L$, and every $\alpha \in (0,1)$,

$$P \succsim Q \iff \alpha P + (1-\alpha) R \succsim \alpha Q + (1-\alpha) R$$

Theorem (vNM)

$\succsim \subset L \times L$ satisfies Axioms 1-3 if and only if there exists $u : X \to \mathbb{R}$ such that, for every $P, Q \in L$

$$P \succsim Q \iff \sum_{x \in X} P(x)u(x) \geqslant \sum_{x \in X} Q(x)u(x)$$

Moreover, in this case u is unique up to a positive linear transformation.

Crucial axiom - independence

- Our version $A_i \sim [A_1, u_i; A_n, 1-u_i] = \tilde{A}_i$

$$(A_1, p_1; ...; A_i, p_i; ...; A_n, p_n) \sim (A_1, p_1; ...; \tilde{A}_i, p_i; ...; A_n, p_n)$$

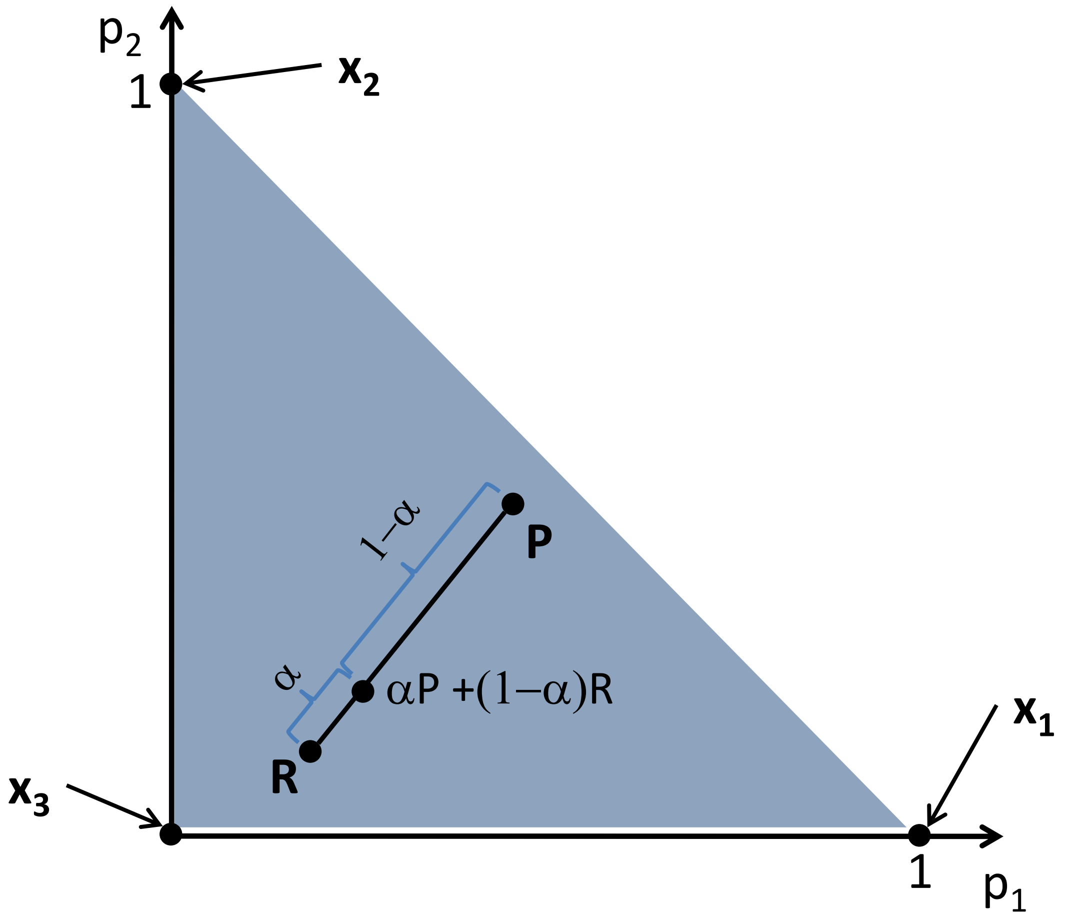
- The general version

For every $P, Q, R \in L$, *and every* $\alpha \in (0,1)$,

$$P \succsim Q \iff \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R$$

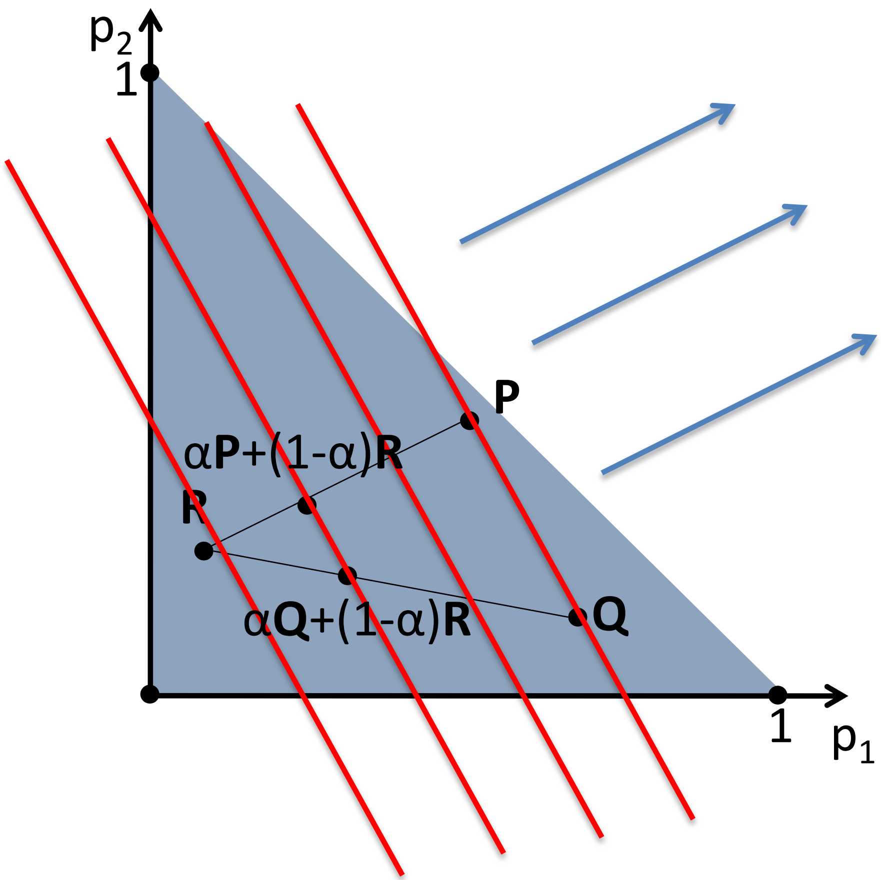
- Why the general version implies our version?

Machina triangle



Independence assumption in the Machina triangle

Suppose that A1 is better than A2 is better than A3



17.1 and 17.2

17.1) Choose one lottery:

P=(1 mln, 1)
Q=(5 mln, 0.1; 1 mln, 0.89; 0 mln, 0.01)

17.2) Choose one lottery:

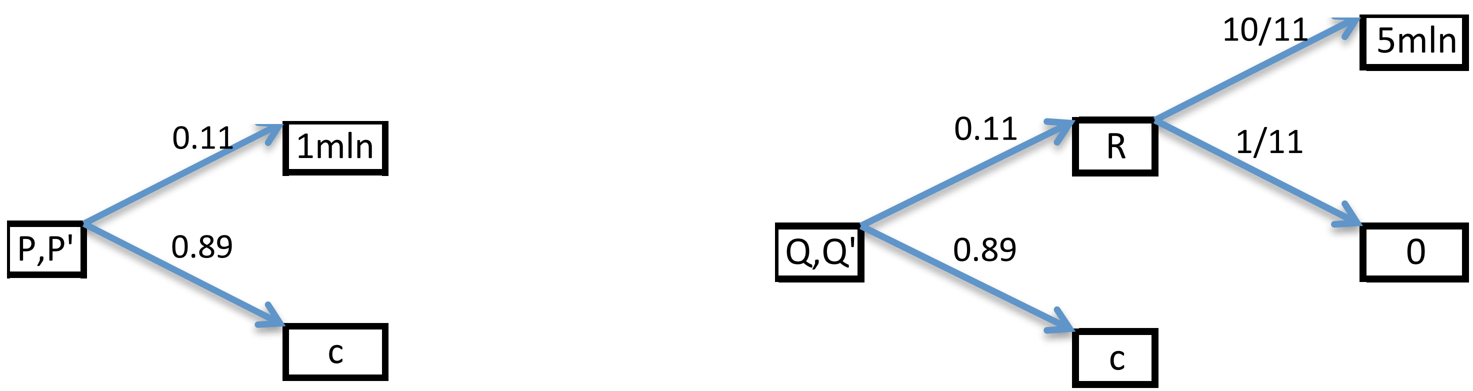
P'=(1 mln, 0.11; 0 mln, 0.89)
Q'=(5 mln, 0.1; 0 mln, 0.9)

Kahneman, Tversky (1979) [**common consequence effect violation of independence**]

Many people choose P over Q and Q' over P'

Common consequence graphically

P = (1 mln, 1)
P'= (1 mln, 0.11; 0, 0.89)
Q = (5 mln, 0.1; 1 mln, 0.89; 0, 0.01)
Q'= (5 mln, 0.1; 0, 0.9)



- If we plug c = 1mln, we get P and Q respectively
- If we plug c = 0, we get P' and Q' respectively

$$P \succsim Q \iff 1\text{mln} \succsim R \iff P' \succsim Q'$$

18.1 i 18.2

18.1) Choose one lottery:

P=(3000 PLN, 1)
Q=(4000 PLN, 0.8; 0 PLN, 0.2)

18.2) Choose one lottery:

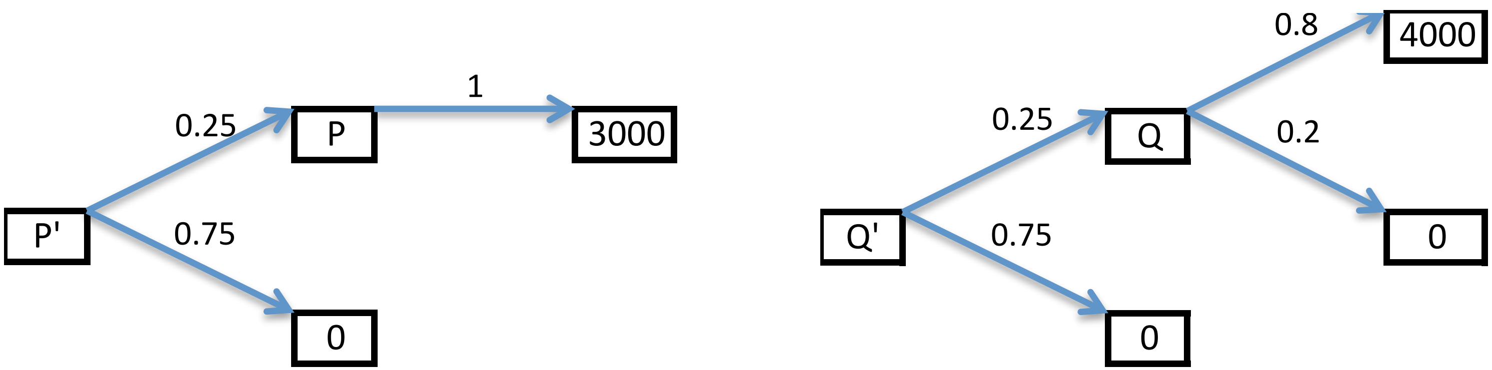
P'=(3000 PLN, 0.25; 0 PLN, 0.75)
Q'=(4000 PLN, 0.2; 0 PLN, 0.8)

Kahneman, Tversky (1979) [**common ratio effect, violation of independence**]

Many people choose P over Q and Q' over P'

Common ratio graphically

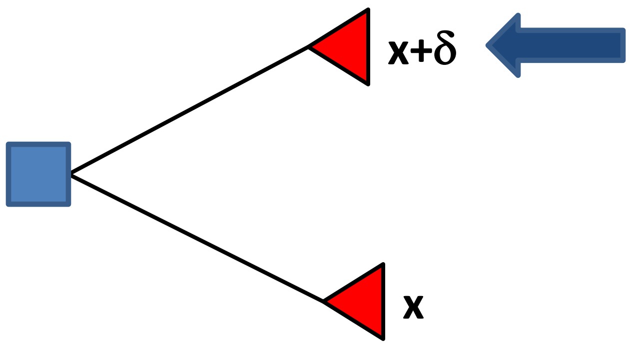
P=(3000 PLN, 1)
P'=(3000 PLN, 0.25; 0 PLN, 0.75)
Q=(4000 PLN, 0.8; 0 PLN, 0.2)
Q'=(4000 PLN, 0.2; 0 PLN, 0.8)



$$P \succsim Q \iff \alpha P + (1-\alpha)0 \succsim \alpha Q + (1-\alpha)0 \iff P' \succsim Q'$$

Monotonicity of utility function

Behaviour	Prefers more to less



Monotonicity of utility function

Behaviour	Prefers more to less
Utility function	$\forall$x, u'(x)>0; u(x) – increasing

$$u(x+\delta) \approx u(x) + \delta u'(x)$$

Monotonicity of utility function

Behaviour	Prefers more to less
Utility function	∀x, u'(x)>0; u(x) – increasing
Attitude towards risk (quantitatively)	u(x)/u'(x) – *fear of ruin*

- prefers more to less
- current wealth x
- probability p of bankruptcy (u(0)=0)
- how much to pay to avoid it?

$$(1-p)u(x)+pu(0)=u(x-d)$$

$$(1-p)u(x)=u(x-d)$$

$$(1-p)u(x)\approx u(x)-du'(x)$$

$$d\approx\frac{u(x)}{u'(x)}p>0$$

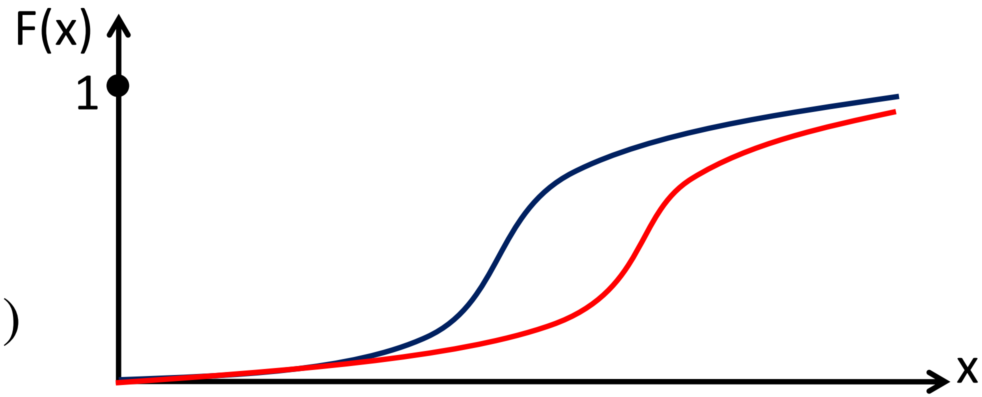
Monotonicity of utility function

Behaviour	Prefers more to less
Utility function	$\forall$x, u'(x)>0; u(x) – increasing
Attitude towards risk (quantitatively)	u(x)/u'(x) – *fear of ruin*
When is the choice obvious	First order stochastic dominance (comparing cdf) – FOSD

A FOSD B :

$$\underset{x}{\forall} F_A(x) \le F_B(x) \wedge \underset{x}{\exists} F_A(x) < F_B(x)$$

$$\text{A FOSD B} \Rightarrow E_A(u(x)) > E_B(u(x))$$

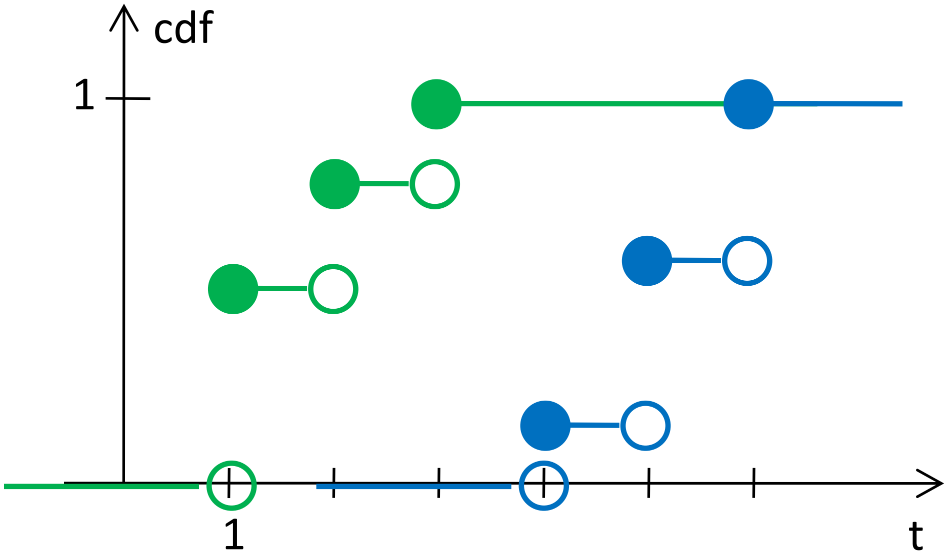


$$\int u(x) f(x) dx = u(x)F(x) - \int u'(x)F(x)dx$$

First Order Stochastic Dominance (FOSD)

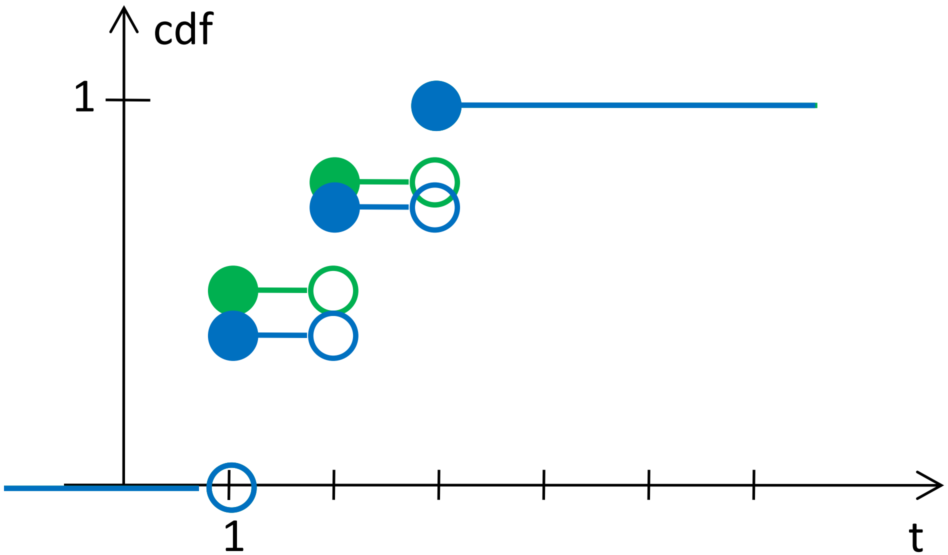
Payoff	Pr.
1	50%
2	30%
3	20%

Payoff	Pr.
4	10%
5	50%
6	40%



Payoff	Pr.
1	50%
2	30%
3	20%

Payoff	Pr.
1	40%
2	35%
3	25%



FOSD

- Assume X and Y are two different lotteries (F, G are not the same)
- Lottery **X FOSD Y** if:

For all a, $\Pr[X > a] \geq \Pr[Y > a]$

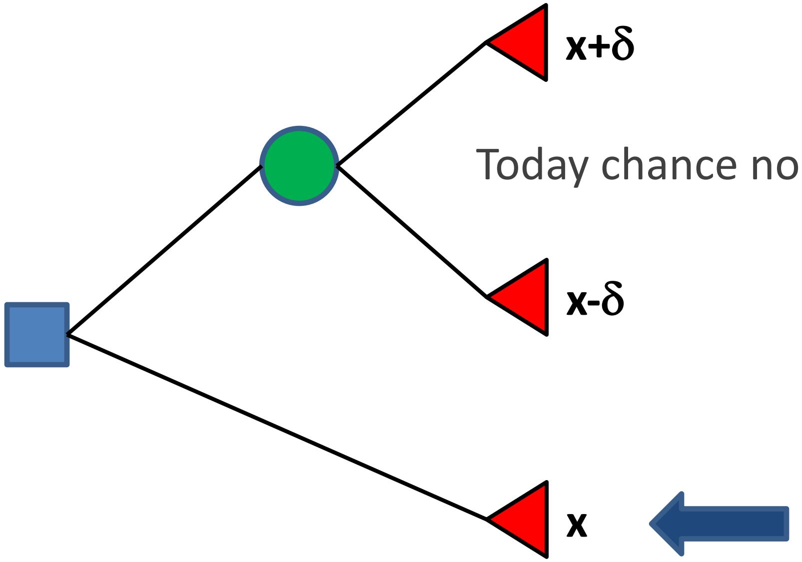
hence: $\boldsymbol{G(a) - F(a) \geq 0}$

Those who prefer more to less will never choose lottery that is dominated in the above sense.

- **Theorem: X FOSD Y if and only if Eu(X) ≥ Eu(Y) holds for all inreasing u**

Marginal utility

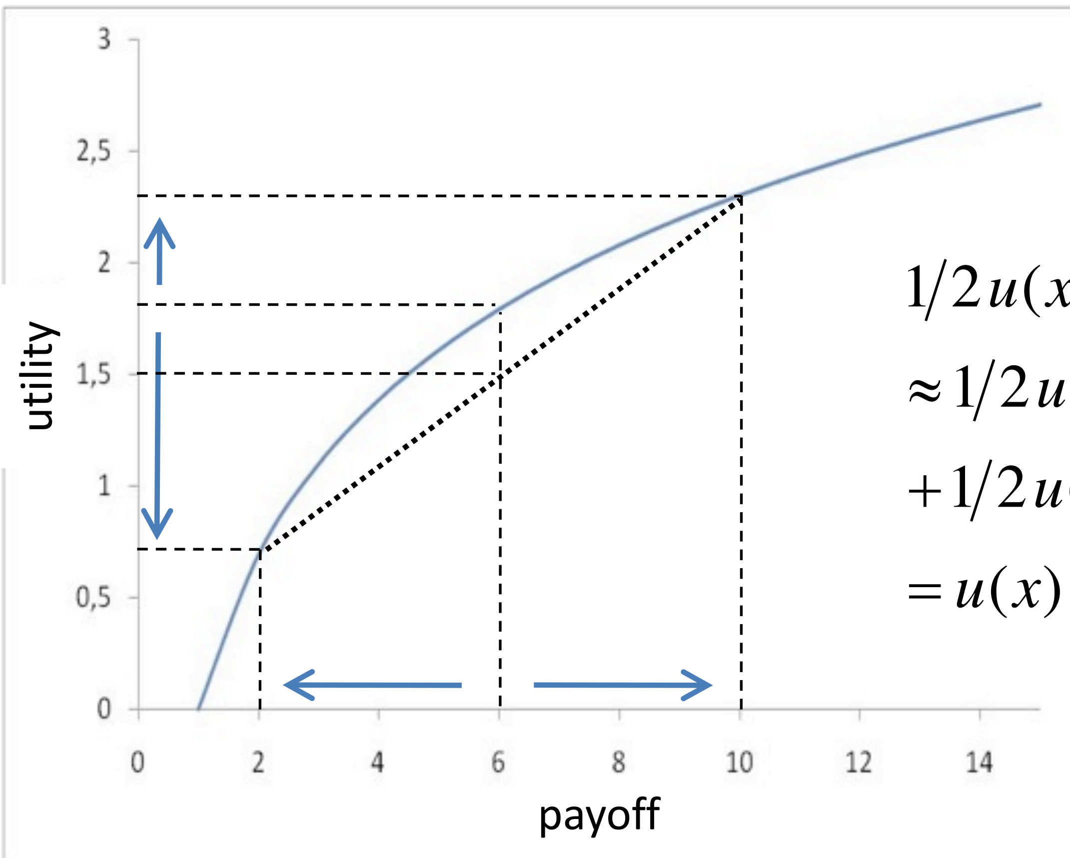
Behaviour	Risk averse, ie. Var(L)>0 $\Rightarrow$ u(E(L))>E(u(L))



Today chance nodes split 50:50

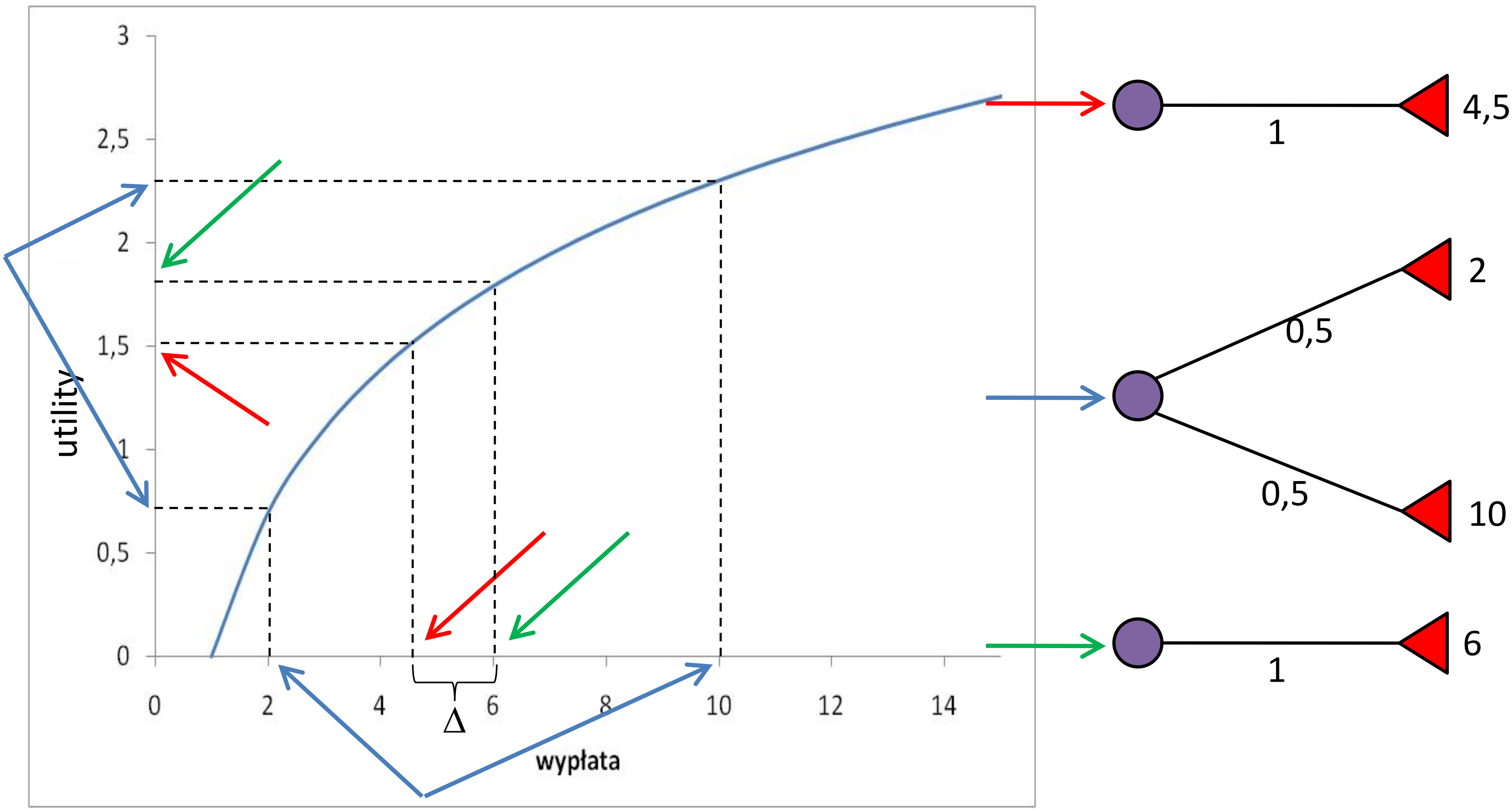
Marginal utility

Behaviour	Risk averse, ie. Var(L)>0 $\Rightarrow$ u(E(L))>E(u(L))
Utility function	$\forall$x, u'(x)>0, u''(x)<0; u(x) – concave, increasing



$$1/2\,u(x+\delta)+1/2\,u(x-\delta)\approx$$
$$\approx 1/2\,u(x)+1/2\,\delta u'(x)+1/4\,\delta^2 u''(x)+$$
$$+1/2\,u(x)-1/2\,\delta u'(x)+1/4\,\delta^2 u''(x)=$$
$$=u(x)+1/2\,\delta^2 u''(x)<u(x)$$

Certainty equivalent and risk premium



Marginal utility

Behaviour	Risk averse, ie. Var(L)>0 $\Rightarrow$ u(E(L))>E(u(L))
Utility function	$\forall$x, u’(x)>0, u’’(x)<0; u(x) – concave, increasing
Attitude towards risk (quantitatively)	-u’’(x)/u’(x) – Arrow-Pratt risk aversion coeff.

- *x* – initial wealth (number)
- *l* – lottery with zero exp. value (random variable)
- *k* – multiplier (we tak k close to zero)
- *d* – risk premium (number)
- *x-d* – certainty equivalent for x+kl

$$\mathrm{E}(u(x-d)) = \mathrm{E}(u(x+kl))$$

$$\mathrm{LHS} \approx \mathrm{E}(u(x) - du'(x)) = u(x) - du'(x)$$

$$\mathrm{RHS} \approx \mathrm{E}(u(x) + klu'(x) + \frac{k^2 l^2}{2} u''(x)) = u(x) + u'(x)\mathrm{E}(l) + \frac{u''(x)}{2} \mathrm{k}^2 \mathrm{E}(l^2)$$

$$-du'(x) = \frac{u''(x)}{2} \mathrm{k}^2 \mathrm{E}(l^2) \Rightarrow d = -\frac{u''(x)}{u'(x)} \times \frac{1}{2} \mathrm{k}^2 \mathrm{D}^2(l)$$

Marginal utility

Behaviour	Risk averse, ie. Var(L)>0 $\Rightarrow$ u(E(L))>E(u(L))
Utility function	$\forall$x, u'(x)>0, u''(x)<0; u(x) – concave, increasing
Attitude towards risk (quantitatively)	-u''(x)/u'(x) – Arrow-Pratt risk aversion coeff.
When is the choice obvious	Second order stochastic dominance (integrals of cdf) – SOSD

$$\text{A SOSD B}:$$

$$\forall_x \int_{-\infty}^{x} F_A(t)dt \leq \int_{-\infty}^{x} F_B(t)dt \wedge \exists_x \int_{-\infty}^{x} F_A(t)dt < \int_{-\infty}^{x} F_B(t)dt$$

$$\text{A SOSD B} \Rightarrow E_A(u(x)) > E_B(u(x))$$

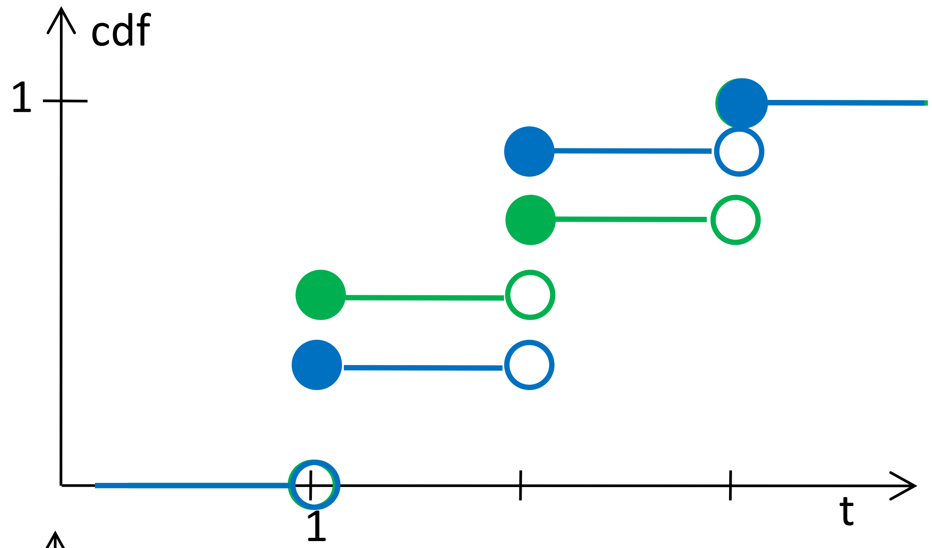
Second Order Stochastic Dominance

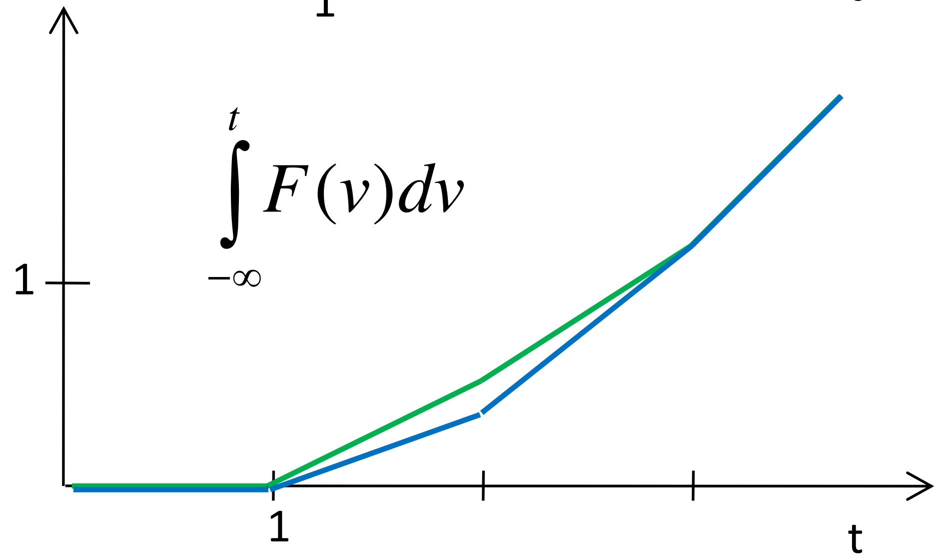
Payoff	Pr.
1	30%
2	60%
3	10%

Payoff	Pr.
1	50%
2	20%
3	30%

t	F(x)	Sum
1	0,3	0
2	0,9	0,3
3	1	1,2
4	1	2,2

t	F(x)	Sum
1	0,5	0
2	0,7	0,5
3	1	1,2
4	1	2,2





SOSD

- Assume X and Y are two different lotteries (F, G are not the same)
- Lottery **X SOSD Y** if:

For all a

$$\int_{-\infty}^{a} \Pr[X > t]\, dt \geq \int_{-\infty}^{a} \Pr[Y > t]\, dt$$

hence: $$\int_{-\infty}^{\boldsymbol{a}} [\boldsymbol{G}(\boldsymbol{t}) - \boldsymbol{F}(\boldsymbol{t})]\boldsymbol{dt} \geq \mathbf{0}$$

- Those who are risk averse will never choose a lottery that is dominated in the above sense.
- **Theorem: X SOSD Y if and only if Eu(X) ≥ Eu(Y) holds for all inreasing and concave u**

Mean-variance criterium

- Risk aversion doesn’t mean that always: A better than B, if only E(A)=E(B) and Var(A)<Var(B)
  - some lotteries do not result from another with mean preserving spread
  - is true when Var(A)=0

- Mean-variance criterium works e.g. for normally distributed random variables (lotteries)

	Lottery X			Lottery Y		
Prob.	**x**	**u(x)=ln(x)**	**$(x-EX)^2$**	**y**	**u(y)=ln(y)**	**$(y-EY)^2$**
20%	20,1	3	65,61	4	1,386	64
80%	9,975	2,3	4,1	14	2,639	4
mean	**12**	**2,44**	**16,4**	**12**	**2,388**	**16**

Measures of risk aversion

- Risk premium measures risk aversion with respect to a given lottery
- As a function of payoff values risk aversion is measured by Arrow, Pratt measures of (local) risk aversion

$$\text{ARA} = -\frac{u''(x)}{u'(x)}$$

$$\text{RRA} = -\frac{u''(x)x}{u'(x)}$$

Exercise 1

- From now on let's assume X is a set of monetary pay-offs (decision maker prefers more money than less)
- Decision maker with vNM utility function prefers lottery (100, ¼; 1000, ¾) to (500, ½; 1000, ½)
- What is a realtion between (100, ½; 500, ¼; 1000, ¼) and (100, ¼; 500, ¾)?
- Suggestion – we can arbitrarily set utility function for two outcomes

Exercise 2

- Decision maker is indifferent between pairs of lotteries:
  (500, 1) and (0, 0,4; 1000, 0,6)
  and
  (300, 1) and (0, ½; 500, ½)

- Can we guess the preference relation between
  (0, 0,2; 300, 0,3; 1000, ½) and (500, 1)?

Exercise 3

- Decision maker with vNM utility is risk averse and indifferent between the following pairs
  (400, 1) and (0, 0,3; 1000, 0,7)
  and
  (0, ½; 200, ½) and (0, 5/7; 400, 2/7)
- Can we guess the preference relation between (200, ½; 600, ½) and (0, 4/9; 100, 5/9)?
- (suggestion – remamber than vNM utility is concave)