Individual decision making under risk

St. Petersburg paradox

- Consider a gamble x in which one of n outcomes will occur
- And these n outcomes are worth respectively $x_1, x_2, ..., x_n$ dollars
- The respective probabilities of these outcomes are $p_1, p_2, ..., p_n$, where each $p_i \in [0, 1]$ and $\sum_{i=1}^{n} p_i = 1$
- **How much is it worth to participate in this gamble?**
- The monetary expected value is $Ex = \sum_{i=1}^{n} x_i p_i$
- One argument says it is the fair price of a gamble.
- **D. Bernoulli - St. Petersburg paradox**:
  - A fair coin is toosed until a head appears.
  - You receive 2^n dollars if the first head occurs on trial n
  - How much are you willing to pay for such gamble?
  - What is the expected value of this gamble?

Bernoulli's resolution

- Expected value of this gamble is infinite, and people rarely want to give more than 10$ for it.
- Bernoulli suggested the following modification to expected value principle.
- Instead of monetary value he proposed to use **intrinsic worths of these monetary values**.
- Intrinsic worth of money increases with money, but at a diminishing rate.
- A function with this property is the **logarithm**
- If the "utility" of m dollars is $\log m$, then the fair price would not be monetary expected value but the monetary equivalent of the utility expected value:
  - **Utility expected value** $\mathrm{E} \log \mathbf{x} = \frac{1}{2} \log 2 + \frac{1}{4} \log 4 + ...$
  - Monetary equivalent $CE(\mathbf{x}) = \exp(\mathrm{E} \log \mathbf{x})$

Criticism of Bernoulli's proposal

There are certain obvious criticism of Bernoulli's tack:

- The utility association to money is completely *ad hoc*.
  - There are many strictly increasing concave functions.
- Why should a decision be based upon the expected value of these utilities?
  - Rationale for expected value is usually based on the effect of repreating the gamble many times.
  - Why should it extend to the situation in which a gamble is played only once?

Von Neumann and Morgenstern

- ▶ What we want is a construction of a utility function for each individual which in some sense represents her choices among gambles.
- ▶ And which has as a consequence the fact that the expected value of utility represents the utility of the corresponding gamble
- ▶ Von Neumann and Morgenstern have shown the following:

Theorem

*If a person is able to express preferences between every possible pair of gambles, where the gambles are taken over some basic set of laternatives, then one can introduce utility associations to the basic alternatives in such a manner that, if the person is guided solely by the utility expected value, **he is acting in accord with his true tastes** - provided only that there is an element of consistency in his tastes.*

vNM utility function

Two points about this theorem and what is called von Neumann Morgenstern utility function:

a) Reflects preferences about the alternatives in a certain given situation.
  - e.g. the resulting function will incorporate attitude towards the whole gambling situation.

b) Justifies the central role of expected value without any further argument
  - Specifically, without any discussion of long run effects.

The essence of the idea

- Suppose that an individual reveals: $A \succ B$, $B \succ C$ and $A \succ C$.
- Ordinal utility - any three numbers in decreasing in magnitude
- But here we admit gambles
- Suppose we ask his preferences between:
  - obtaining B for certain
  - a gamble $(A, p; C, 1 - p)$
- If p is sufficiently close to 1, then a gamble will be preferred
- If p is near 0, then the certain option will be preferred
- There has to be p, such that he is indifferent between the two
- We suppose that there is exactly one such $p \in (0, 1)$, let it be $\frac{2}{3}$

The essence of the idea

- If we arbitrarily associate 1 to alternative A and 0 to C
- Then what number should we associate to B?
- Naturally, $\frac{2}{3}$
- If this choice is made, then B is **fair equivalent** for the gamble $(A, \frac{2}{3}; C, \frac{1}{3})$
- in the sense that the utility of B equals the utility expected value of the gamble

$$1 \times \frac{2}{3} + 0 \times \frac{1}{3} = \frac{2}{3}$$

- There are triples of numbers other than $(1, \frac{2}{3}, 0)$ which also reflect the same preferences as

$$(a + b, \frac{2}{3}a + b, b), \quad \text{where} \, a > 0$$

- Nevertheless, we are not allowed to say that going from B to C is twice as desirable than going from A to B

Consistency demands

1) Any two alternatives should be **comparable**
2) Preference relations for lotteries are **transitive**
3) If a lottery has as one of its alternatives another lottery, then the first lottery is **decomposable** into the more basic alternatives through the use of the probability calculus
4) If two lotteries are indifferent to the subject, then they are **interchangable** as alternatives in any compound lottery
5) If two lotteries involve the same two alternatives, then the one in which the **more preferred altenative has a higher probability of occuring is itself preferred**
6) If A is preferred to B and B to C, then **there exists a lottery involving A and C** (with apropriate probabilities) which is indifferent to B

Formal postulates/assumptions/axioms

- All loteries are build up from a finite set of basic alternatives $A_1, A_2, ..., A_n$
- We assume w.l.o.g. that $A_1 \succsim ... \succsim A_n$ and $A_1 \succ A_n$.
- $L = (A_1, p_1; ...; A_n, p_n)$, where $0 \geqslant p_i \geqslant 1$ and $\sum_{i=1}^n p_i = 1$ is a typical lottery
- Objective probabilities and yet the lottery played once only (we do not view lottery from a frequency point of view)

Assumption (1)
The weak preference relation $\succsim$ defined over the space of basic alternatives is complete and transitive

Reduction of compound lotteries

Assumption (2 reduction of compound lotteries)

If $L^{(i)} = (A_1, p_1^{(1)}; ...; A_n, p_n^{(n)})$, for $i = 1, 2, ..., s$, then

$$(L^{(1)}, q_1; ...; L^{(s)}, q_s) \sim (A_1, p_1; ...; A_n, p_n)$$

where $p_i = q_1 p_i^{(1)} + ... + q_s p_i^{(s)}$.

- ▶ Experiments such as $\mathbf{p}^{(1)}$ and $\mathbf{q}$ are statistically independent
- ▶ Or numbers such as $p_j^{(i)}$ actually denotes the conditional probability of prize j in experiment $\mathbf{p}^{(i)}$ given that lottery i arose from experiment q
- ▶ The assumption abstracts away all "joy in gambling", "atmosphere of the game", "pleasure in suspense"

IIR

Assumption (3 continuity)

For each A_i, there exists a number u_i such that
$A_i \sim [A_1, u_i; A_n, 1 - u_i] = \tilde{A}_i.$

- ▶ Critics emphasize extreme events such as death
- ▶ But it is not really a problem

Assumption (4 Substitutibility or IIA)

For any lottery L:

$$(A_1, p_1; ...; A_i, p_i; ...; A_n, p_n) \sim (A_1, p_1; ...; \tilde{A}_i, p_i; ...; A_n, p_n)$$

- ▶ This assumption, taken with the third, says that not only A_i is indifferent to $\tilde{A}_i$ as taken alone, but also when substituted in any lottery ticket.

Two final assumptions

Assumption (5 transitivity)
Preference among lottery tickets is a transitive relation

Assumption (6 monotonicity)
A lottery $(A_1, p; A_n, 1 - p)$ is weakly preferred to
$(A_1, p'; A_n, 1 - p')$ if and only if $p \geqslant p'$

- But a mountain climber prefers "life" to "death" and also some lottery of life and death to life itself
- However the real alternative is the whole gestalt of climbing
- Sometimes we need richer set of alternatives for this assumption to be valid

Expected Utility Theorem

Theorem

If prefererence relation $\succsim$ satisfies assumptions 1-6, then there are numbers u_i associated with the basic prizes A_i such that for two lotteries L and L' the magnitudes of the expected values

$$p_1 u_1 + ... + p_n u_n \quad \text{and} \quad p'_1 u_1 + ... + p'_n u_n$$

reflect the preference between the lotteries.

Example

- Let $L = (A_1, 0.25; A_2, 0.25; A_3, 0.25; A_4, 0.25)$ and $L' = (A_1, 0.15; A_2, 0.50; A_3, 0.15; A_4, 0.20)$
- Which one should I choose?
- Suppose we determine that $A_2 \sim (A_1, 0.6; A_4, 0.4)$ and $A_3 \sim (A_1, 0.2; A_n, 0.8)$

Proof

1. Let $L = (A_1, p_1; ...; A_n, p_n)$ and $L' = (A_1, p_1'; ...; A_n, p_n')$ be any two lotteries.

2. By Assumption (2) we can always write lotteries in this basic way

3. Replace each A_i in lottery L by $\tilde{A}_i$. Assumption (3) says these indifferent elements exist and assumption (4) says they are substitutable.

4. By using transitvity of indifference serially, we get

$$(A_1, p_1; ...; A_n, p_n) \sim (\tilde{A}_1, p_1; ...; \tilde{A}_n, p_n)$$

5. If we sequentially apply assumption (2), we get:

$$(A_1, p_1; ...; A_n, p_n) \sim (A_1, p; A_n, 1 - p)$$

where $p = p_1 u_1 + ... + p_n u_n$

6. Do steps 3-5 for lottery L' to arrive at
$p' = p_1' u_1 + ... + p_2' u_2$.

7. Finally, by assumption (6) $L \succsim L' \iff p \geqslant p'$

Ordinal vs cardinal utility

Lottery	A_1	A_2	A_3	A_4
u	1.0	0.6	0.2	0.0
u'	1.6	0.8	0.0	-0.4

Lottery	A_1	A_2	A_3	A_4
v	1.0	0.6	0.2	0.0
v'	16	15.9	3.0	2.0

Lottery	A_1	A_2	A_3	A_4
s	1.0	0.6	0.2	0.0
s'	3.0	4.0	1.0	2.2

- **Cardinal** von Neumann Morgenstern utility function is **unique up to affine transformation**
- **Ordinal** utility function is **unique up to any strictly monotonic transformation**

Some common falacies

- The decision maker behaves as if he were a maximizer of expected values of utility

Some common falacies

Falacy (1)

Lottery $(A_1, p_1; ...; A_n, p_n)$ is preferred to lottery $(A_1, p'_1; ...; A_n, p'_n)$ because the utility of the former, $p_1 u_1 + ... + p_n u_n$, is greater than the utility of the latter, $p'_1 u_1 + ... + p'_n u_n$.

- We only devise a convenient way to represent preferences
- Preferences among lotteries logically precede the introduction of utility function

Some common falacies

Falacy (2)

Suppose that $A \succ B \succ C \succ D$ and that the utilities of these alternatives satisfy $u(A) + u(D) = u(B) + u(C)$, then $(B, \frac{1}{2}; C, \frac{1}{2})$ should be preferred to $(A, \frac{1}{2}; D, \frac{1}{2})$, because although they have the same expected utility, but the former has smaller variance.

- Utility function reflects completely a person's preferences among risky alternatives

Falacy (3)

Suppose that $A \succ B \succ C \succ D$ and that the utilities of these alternatives satisfy $u(A) - u(B) > u(C) - u(D)$, then the change from B to A is more preferred than the change from D to C

- Utility function is constructed from preferences between pairs of alternatives, not between pairs of pairs of alternatives

Some common falacies

Falacy (4)

Interpersonal comparisons of utility

- ▶ Utility function is unique up to affine transformation
- ▶ We can choose a zero and a unit as we please
- ▶ Zero is not a problem but **arbitrary unit** is
- ▶ Suppose two people are isolated from each other and each is given a measuring stick with some (possibly different) unit of measurement
- ▶ One subject is given full scale plans of a building
- ▶ He is permitted to send only angles and lengths (in his units of measurement) to the other guy

Conclusion: There is no natural zero in utility measurement.

- In the ordinal utility representation for choice under certainty the crucial axiom "giving" order is TRANSITIVITY.
- Completeness is more a "technical" assumption needed in order to be able to make comparisons
- Here in the cardinal representation for choice under risk the crucial axiom "giving" cardinality is INDEPENDENCE
- Continuity is merely a technical assumption needed for existence of utility associations
- Monotonicity (which gives uniqueness of utility associations) and reduction of compound lotteries are not needed since they follow from other axioms in the general case
- On the next slide the more general formulation is presented

Axioms of von Neumann and Morgenstern

Axiom (Weak order)

$\succsim$ *is complete and transitive*

Axiom (Continuity)

For every $P, Q, R \in L$,

$$P \succ Q \succ R \implies \exists \alpha, \beta \in (0,1) : \alpha P + (1-\alpha)R \succ Q \succ \beta P + (1-\beta)R$$

Axiom (Independence)

For every $P, Q, R \in L$, *and every* $\alpha \in (0,1)$,

$$P \succsim Q \iff \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R$$

Expected Utility Theorem

Theorem (vNM)

$\succsim \subset L \times L$ *satisfies Axioms 1-3 if and only if there exists*
$u : X \to \mathbb{R}$ *such that, for every* $P, Q \in L$

$$P \succsim Q \iff \sum_{x \in X} P(x)u(x) \geqslant \sum_{x \in X} Q(x)u(x)$$

Moreover, in this case u *is unique up to a positive linear transformation.*