Mark Machina (UCSD) - our "publicity director" who asked about monotonicity and invented the triangle

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Drazen Prelec (MIT) who likes Miłosz poetry and who discovered the most famous shape of the probability weighting function

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Definition

The CE ***functional is monotonic wrt FOSD*** *if whenever* $x \succ_{FOSD} y$, $\mathrm{CE}(x) > \mathrm{CE}(y)$.

Definition

The CE ***functional is continuous*** *if for every sequence of lottery payoffs* $\{x_n\}$*, where* $n \in \mathbb{N}$ *and each* x_n *is distributed according to* F_n*, converging in distribution to the lottery payoff* y *distributed according to* G*, the following holds:* $\lim_{n \to \infty} \mathrm{CE}(x_n) = \mathrm{CE}(y)$.

Monotonicity and continuity in the decision utility model

Define: $C(\eta) = 1 - D(1-\eta)$, $\eta \in [0,1]$. And then also
$\mathrm{RRA}_D(\eta) = -\frac{\eta D''(\eta)}{D'(\eta)}$, $\mathrm{RRA}_C(\eta) = -\frac{\eta C''(\eta)}{C'(\eta)}$, $\eta \in [0,1]$

Theorem (Monotonicity and Continuity)

1) *The CE **functional is monotonic wrt FOSD** if and only if* RRA_D *and* RRA_C *are non-decreasing.*
2) *The CE **functional is continuous** if and only if D is linear.*
   a) *Continuity wrt. upper range increase holds if and only if* RRA_D *is constant (power function).*
   b) *Continuity wrt. lower range increase holds if and only if* RRA_C *is constant (inverse power function).*

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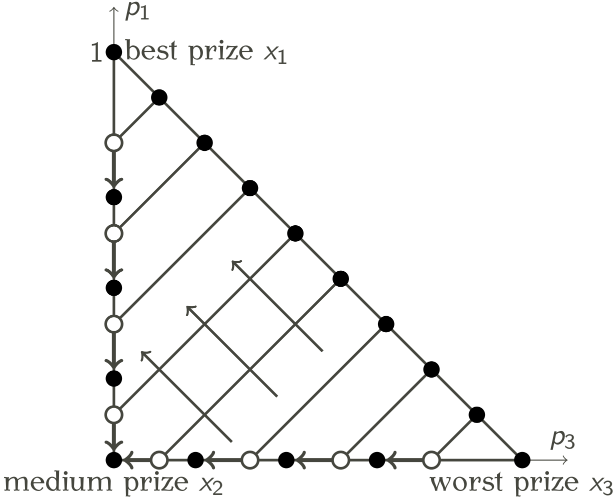
Indifference lines for the decision utility satisfying monotonicity



Example 1: The CDF of the Beta distribution

$$D(x) = A\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt,$$

where $x \in [0,1]$, $A = \frac{1}{\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt}$, and $\alpha, \beta > 0$.

Monotonicity conditions are satisfied in four special cases:

a) **linear**: $D(x) = x$, $\alpha = \beta = 1$,

b) **concave inverse power**: $D(x) = 1-(1-x)^\beta, \beta > 1$, $\alpha = 1$,

c) **convex power**: $D(x) = x^\alpha, \alpha > 1$, $\beta = 1$,

d) **all S-shaped functions in this family**, $\alpha, \beta > 1$.

Example 2: The CDF of the Two-Sided Power Distribution

$$D(x) = \begin{cases} x_0 \left(\frac{x}{x_0}\right)^{\alpha}, & 0 \leqslant x \leqslant x_0, \\ 1-(1-x_0)\left(\frac{1-x}{1-x_0}\right)^{\alpha}, & x_0 \leqslant x \leqslant 1, \end{cases}$$

where $x_0 \in (0,1)$, $\alpha > 0$.

Monotonicity conditions are satisfied in four special cases:

a) **linear**: $D(x) = x$, $\alpha = 1$,

b) **concave inverse power**: $D(x) = 1-(1-x)^{\alpha}, \alpha > 1$, $x_0 = 0$,

c) **convex power**: $D(x) = x^{\alpha}, \alpha > 1, x_0 = 1$,

d) **all S-shaped functions in this class**, $\alpha > 1$, $x_0 \in (0,1)$.

All inverse S-shaped functions in both classes are excluded.

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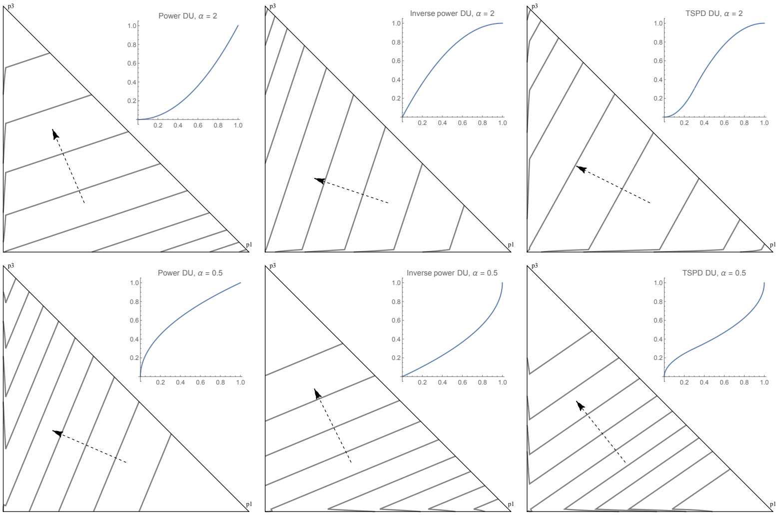
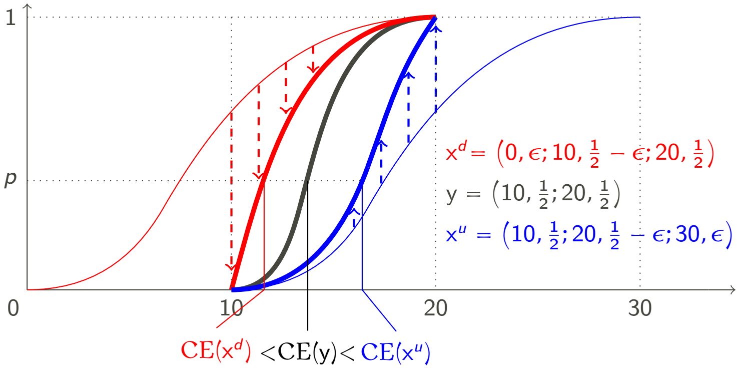
Indifference lines for TSPD decision utilities



Monotonicity and continuity for S-shaped functions

From now on let $\mathrm{CE}(x^d)$, $\mathrm{CE}(x^u)$ denote the limits as $\epsilon \to 0^+$.



- **Continuity is generally violated** in the decision utility model
- Monotonicity is **typically satisfied for S-shaped** fcns
- Monotonicity is **always violated for inverse S-shaped** fcns

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Monotonicity and continuity for the limiting functions

limiting functions	$D(x)$	$\mathrm{CE}(x^d)$	$\mathrm{CE}(y)$	$\mathrm{CE}(x^u)$
convex power	x^2	15.81	17.07	17.07
concave power	$\sqrt{x}$	14.57	12.5	12.5
convex inverse power	$1-\sqrt{1-x}$	17.5	17.5	15.43
concave inverse power	$1-(1-x)^2$	12.93	12.93	14.81

- Power is continuous wrt upward range changes
- Inverse power is continuous wrt downward range changes
- Concave power and convex inverse power violate monotonicity
- Convex power and concave inverse power satisfy monotonicity

Coexistence of gambling and insurance:

$$(J - pJ, p; -pJ, 1-p) \succ (0, 1),$$
$$(H, 1-p; 0, p) \prec (H - pH, 1).$$

This pattern of preferences is predicted by the decision utility model if the following conditions are satisfied:

$$\boxed{p > \max(D(p), 1 - D(1-p))}$$

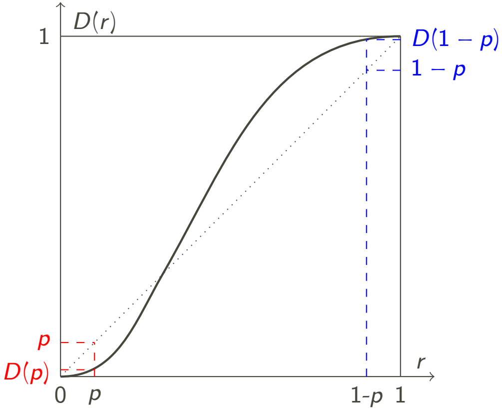

Figure: gambling – no gambling and insurance – no insurance comparison.

- binary lotteries: DU is observationally equivalent to DT
- However psychologically very different, based on an S-shaped utility function and hence much closer to Markowitz (1952)

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Harry Markowitz (La Jolla) who is more proud of his von Neumann prize for his work on utility rather than his Nobel prize for his work on optimal portfolio

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Russian roulette

Two situations:

1. A six-shooter with 4 loaded chambers. How much would you pay to remove one bullet?
2. A six-shooter with 2 loaded chambers. How much would you pay to remove two bullets?

Expected Utility Theory predicts that the two prices should be the same (Assumption: if you die you don't care)

$$\begin{aligned}
\frac{4}{6}u(\text{death}) + \frac{2}{6}u(w) &= \frac{3}{6}u(\text{death}) + \frac{3}{6}u(w-P) \\
\frac{2}{6}u(\text{death}) + \frac{4}{6}u(w) &= u(w-Q)
\end{aligned}$$

Assuming that $u(\text{death}) = 0$ and $u(w) = 1$, we get:

$$u(w-P) = 2/3 = u(w-Q) \quad \Rightarrow \quad P = Q$$

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Russian roulette

Let's see how it is with the Decision Utility model:

$$\text{death} + (w - \text{death})D^{-1}\left(\frac{1}{3}\right) = \text{death} + (w - P - \text{death})D^{-1}\left(\frac{1}{2}\right)$$

$$\text{death} + (w - \text{death})D^{-1}\left(\frac{2}{3}\right) = w - Q$$

Hence we get the following conditions:

$$\frac{D^{-1}\left(\frac{1}{3}\right)}{D^{-1}\left(\frac{1}{2}\right)} = \frac{w - P - \text{death}}{w - \text{death}}$$

$$\frac{D^{-1}\left(\frac{2}{3}\right)}{D^{-1}(1)} = \frac{w - Q - \text{death}}{w - \text{death}}$$

Finally we get:

$$Q > P \iff \frac{D^{-1}\left(\frac{2}{3}\right)}{D^{-1}(1)} < \frac{D^{-1}\left(\frac{1}{3}\right)}{D^{-1}\left(\frac{1}{2}\right)}$$

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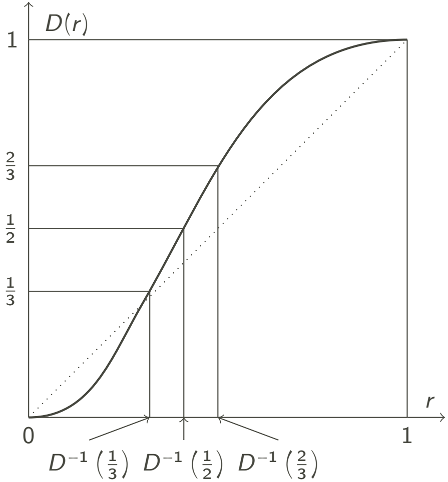
Russian roulette



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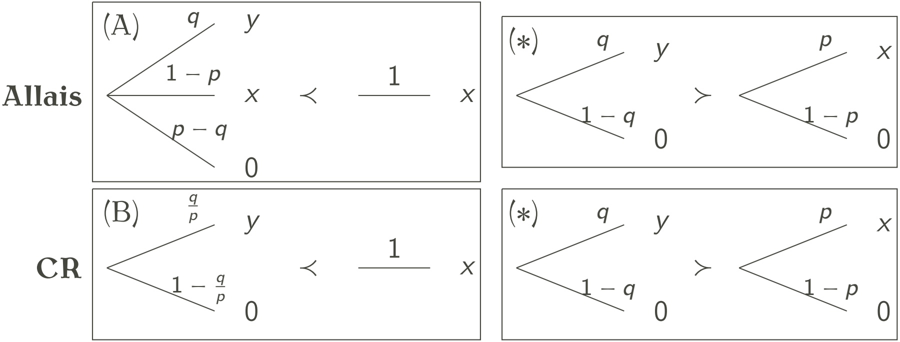
The Allais paradox and the Common Ratio effect

Let $y > x > 0$, $1 > p > q > 0$, $\frac{q}{p} > p$,
e.g. $y = \$4\,000$, $x = \$3\,000$, $q = 0.2$, $p = 0.25$.



EU: (A),(B) equivalent and cannot coexist with (∗).
DU: (A),(B) equivalent and can coexist with (∗).
Rank: (A),(B) not equivalent and can coexist with (∗).

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The Allais paradox and the Common Ratio effect

$$\textbf{EU:}\quad \underbrace{\frac{u(W+x)}{u(W+y)} < \overbrace{\frac{q}{p}}^{}}_{\text{(A),(B)}}\overbrace{< \frac{u(W+x)}{u(W+y)}}^{(*)} \text{ ... contradiction}$$

$$\textbf{DU:}\quad \underbrace{D^{-1}\left(\frac{q}{p}\right) < \frac{x}{y}}_{\text{(A),(B)}} \overbrace{< \frac{D^{-1}(q)}{D^{-1}(p)}}^{(*)} \text{ ... satisfied when } D \text{ is flat in the}$$

upper and steep in the middle part of its domain.

$$\textbf{Rank:}\quad \underbrace{\frac{w(q)}{w(q)+1-w(1-p+q)} < \frac{x}{y}}_{\text{(A)}} \overbrace{< \frac{w(q)}{w(p)}}^{(*)}$$

$$\underbrace{w\left(\frac{q}{p}\right) < \frac{x}{y}}_{\text{(B)}} \overbrace{< \frac{w(q)}{w(p)}}^{(*)}$$

In the slide, each (*) overbrace spans the middle term together with the right-hand inequality ($\frac{q}{p} < \frac{u(W+x)}{u(W+y)}$ and $\frac{x}{y} < \ldots$).

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Predictive accuracy

For binary lotteries, range dependence equivalent to rank dependence.

How about more than two outcome lotteries? Convenient to check in the MM triangle:

Harless (1992) finds that systematic violations of expected utility disappear when lotteries are **nudged inside the triangle**. Similar evidence: Conlisk (1989), Sopher, Gigliotti (1993), Harless, Camerer (1994), Cohen (1992), Hey, Orme (1994).

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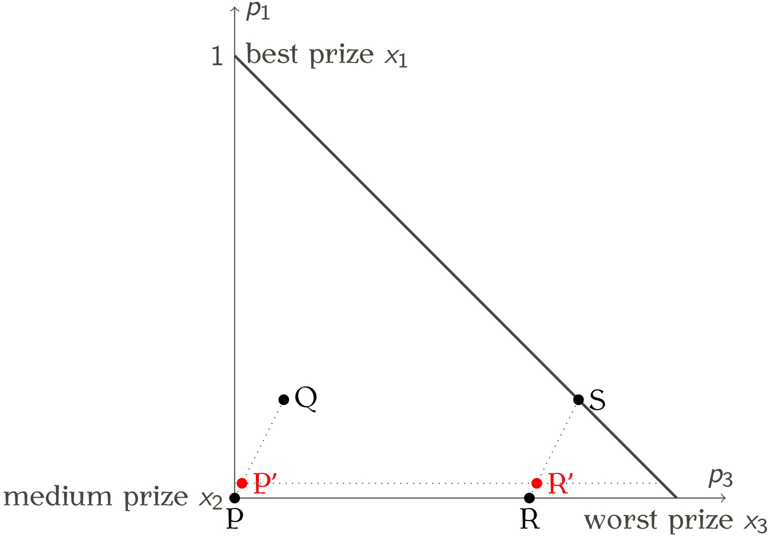
Nudging the lotteries inside the MM triangle: Allais



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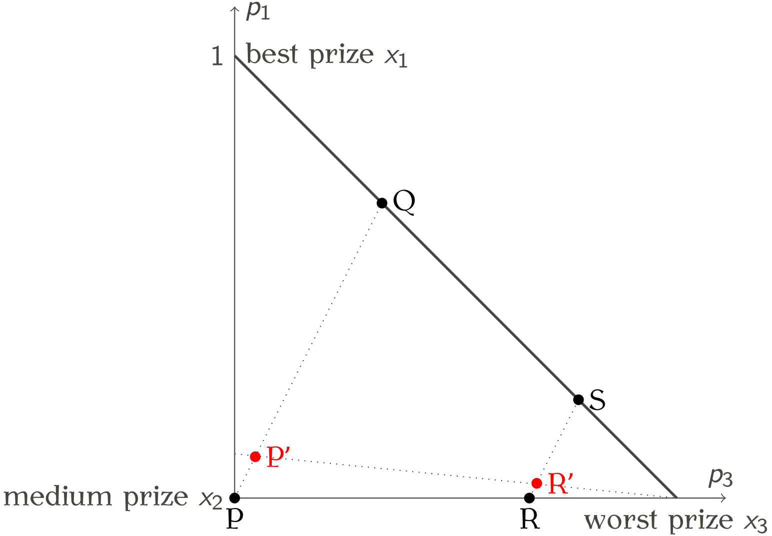
Nudging the lotteries inside the MM triangle: common ratio



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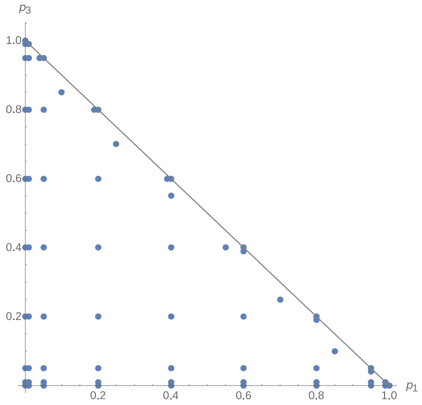
Predictive accuracy: comparison with CPT

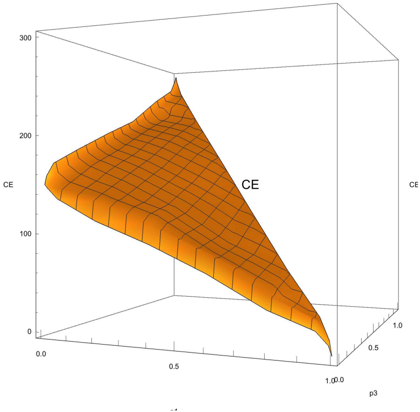
Kontek (2018) nonparametrically fits indifference curves in the MM triangle.

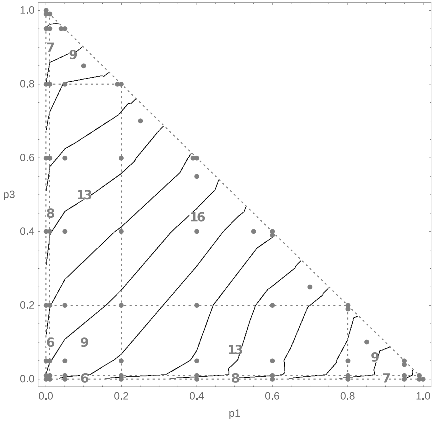
His choice of the grid is novel – more dense on the edges:



Predictive accuracy: comparison with CPT

What he gets is the following fit:





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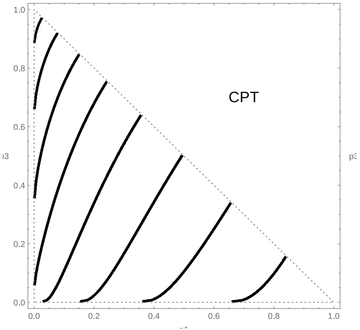
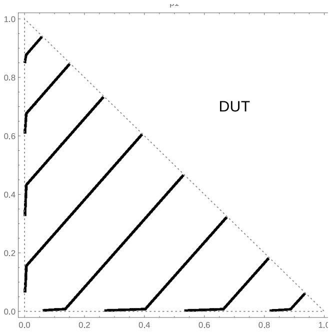
Theoretical predictions of CPT and DUT

CPT predicts smooth nonlinear curves with fanning out.



DUT predicts straight parallel lines discontinuous at the legs.



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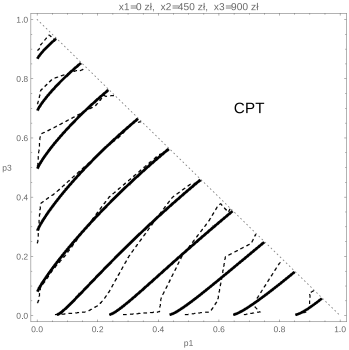
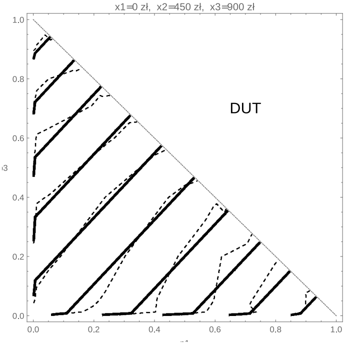
Which is the better match?

CPT against the data



DUT against the data



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Comparing CPT and DUT - numerical results

The result of fitting 134 aggregated (20% trimmed mean) CE values for a group of 237 subjects (undergraduate students):

Table 1: Estimation results of several decision-making models under risk.

				Parameters		
Model	SSE	AIC	BIC	Est. value	St. error	p-value
EV	54 792.9	1190.1	1195.9			
EUT	54 631.6	1189.7	1195.5	$\alpha = 0.99$	0.02	$< 10^{-101}$
ST	46 427.1	1169.9	1178.6	$\delta = 0.91$	0.02	$< 10^{-92}$
				$\theta = 20904$	43400	0.63
CPT	32 118.0	1122.5	1134.1	$\alpha = 1.12$	0.05	$< 10^{-46}$
				$\gamma = 1.09$	0.04	$< 10^{-52}$
				$\delta = 0.86$	0.01	$< 10^{-96}$
TAX	30 183.1	1114.2	1125.8	$\alpha = 1.05$	0.02	$< 10^{-83}$
				$\gamma = 0.95$	0.02	$< 10^{-73}$
				$\delta = 0.12$	0.02	$< 10^{-5}$
PRT	24 860.8	1086.2	1094.9	$\alpha = 0.96$	0.01	$< 10^{-124}$
				$\beta = 0.91$	0.01	$< 10^{-139}$
DUT	20 003.7	1057.1	1065.8	$r_0 = 0.40$	0.02	$< 10^{-37}$
				$\delta = 1.24$	0.02	$< 10^{-105}$

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Range-Dependent Utility

Krzysztof Kontek, Michal Lewandowski

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Manel Baucells (Darden, U Virginia) who accepted our paper in Management Science and started collaborating with Krzysztof and me on extending the model.

Cap sa Sal, Costa Brava

Sopot, Zatoka Gdańska

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(Exponentially) Discounted Utility Theory

The main model for risk is **Expected Utility**. The main model for intertemporal decisions is **Discounted Utility** theory.

$\mathcal{T} = \{0, 1, \ldots, T\}$ the time index set.

$(c_t, c_{t+1}, \ldots, c_T)$, also denoted by (c_t, c_{-t}), consumption streams

$\succsim_t$ the preference relation over such streams

Utility that represents $\succsim_t$ is the following:

$$\mathrm{DU}_t(c_t, c_1, \ldots, c_T) = u(c_t) + \sum_{n=t+1}^{T} \delta^{n-t} u(c_n),$$

where $\delta \in (0, 1)$, u is a strictly increasing instantaneous utility function satisfying $u(0) = 0$.

Discounted utility theory - main properties

- **Impatience** (dislikes the delay of gains)
- **Stationarity** (preferences are invariant to adding common delays in time): for any c, c', t, t', Δ

$$[c, t] \succsim_0 [c', t'] \iff [c, t + \Delta] \succsim_0 [c', t' + \Delta]$$

, where $[c, t]$ denotes a consumption stream where $c_s = 0$ for $s \neq t$ and $c_s = c$ for $s = t$.

- **Separability**:
  - Current separability: for all $c_0, c'_0, \mathsf{c}_{-0}, \mathsf{c}'_{-0}$: $(c_0, \mathsf{c}_{-0}) \succsim_0 (c_0, \mathsf{c}'_{-0}) \iff (c'_0, \mathsf{c}_{-0}) \succsim_0 (c'_0, \mathsf{c}'_{-0})$.
  - Forward separability: for all $c_0, c'_0, \mathsf{c}_{-0}, \mathsf{c}'_{-0}$: $(c_0, \mathsf{c}_{-0}) \succsim_0 (c'_0, \mathsf{c}_{-0}) \iff (c_0, \mathsf{c}'_{-0}) \succsim_0 (c'_0, \mathsf{c}'_{-0})$.
- **Dynamic consistency**: for all t, c_t, c_{-t}, c'_{-t}: $(c_t, \mathsf{c}_{-\mathsf{t}}) \succsim_t (c_t, \mathsf{c}'_{-t}) \iff \mathsf{c}_{-t} \succsim_{t+1} \mathsf{c}'_{-t}$.

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Discounted Utility Theory paradoxes

Evidence against:

- Stationarity: preference reversal due to desire for immediate gratification, e.g.:

$$[100, 0] \succsim_0 [105, 1] \text{ and } [100, 12] \precsim_0 [105, 13]$$

- Separability: Loewenstein, Prelec (1993), 5 weekends, H eat at home, F fancy French, L fancy Lobster:

Group I:	option A:	F, H, H, H, H	[11%]
vs.	option B:	H, H, F, H, H	[89%]
Group II:	option C:	F, H, H, H, L	[49%]
vs.	option D:	H, H, F, H, L	[51%]

- Dynamic consistency: Self control problems, e.g. I will exercise tomorrow

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Hyperbolic or quasi-hyperbolic discounting

Behavioral model for choice over time is quasi-hyperbolic discounting (or beta-delta model):

$$\mathrm{BDU}_t(c_t, c_1, \ldots, c_T) = u(c_t) + \beta \left(\sum_{n=t+1}^{T} \delta^{n-t} u(c_n) \right),$$

Quasihyperbolic approximates a non-tractable hyperbolic case:

discounting/period	0	1	2	$\cdots$	T
exponential	1	δ	δ^2	$\cdots$	δ^T
hyperbolic	1	$\frac{1}{1+k}$	$\frac{1}{1+2k}$	$\cdots$	$\frac{1}{1+Tk}$
quasi-hyperbolic	1	$\beta\delta$	$\beta\delta^2$	$\cdots$	$\beta\delta^T$

The BD model explains nonstationarity and dynamic inconsistency but fails to explain non-separabilities.

Paradoxes for risk and time

Choice objects: (x, p, t), where x is money, p probability, t time delay

Table 1 Choices Between Prospects A and B

	Prospect A	vs.	Prospect B	Response	*N*
1.	**(€9, for sure, now)**	vs.	(€12, with 80%, now)	**58%** vs. 42%	142
2.	(€9, with 10%, now)	vs.	**(€12, with 8%, now)**	22% vs. **78%**	65
3.	(€9, for sure, 3 months)	vs.	**(€12, with 80%, 3 months)**	43% vs. **57%**	221
4.	**(ƒ100, for sure, now)**	vs.	(ƒ110, for sure, 4 weeks)	**82%** vs. 18%	60
5.	(ƒ100, for sure, 26 weeks)	vs.	**(ƒ110, for sure, 30 weeks)**	37% vs. **63%**	60
6.	(ƒ100, with 50%, now)	vs.	**(ƒ110, with 50%, 4 weeks)**	39% vs. **61%**	100
7.	**(€100, for sure, 1 month)**	vs.	(€100, with 90%, now)	**81%** vs. 19%	79
8.	(€5, for sure, 1 month)	vs.	**(€5, with 90%, now)**	43% vs. 57%	79

Sources. Rows 1–3, Baucells and Heukamp (2010, Table 1); rows 4–6, Keren and Roelofsma (1995, Table 1) (ƒ1 in 1995 equaled $0.6); rows 7 and 8, Baucells et al. (2009).

- Pattern 1-2: the common ratio effect
- Pattern 4-5: the common difference effect
- Pattern 1-3: the common ratio using delay
- Pattern 4-6: the common difference using probability
- Pattern 7-8: subendurance

They consider preferences over triplets (x, p, t), which describe a prospect of receiving \$$x$ with probability p in time t, otherwise nothing.

Their idea is to see time as intrinsically uncertain: delaying the receipt of a prize is equivalent to increasing uncertainty of getting it.

They postulate the following axiom which is key in their model:

$$(x, p, t + \Delta) \sim (x, \theta p, t) \implies (x, q, s + \Delta) \sim (x, q\theta, s),$$

for all $(x, p, t), (x, q, s), \Delta > 0, \theta \in (0, 1)$.

Motivation for Range Utility Theory for risk and time

The normative (rational) theory for risk and time is Discounted Expected Utility, $U = \mathbb{E}[\exp(-\rho t) u(X_t)]$

We have good descriptive (behavioral) theories, but ONLY for

- Gambles that **resolve today**, e.g. prospect theory
- Streams of positive outcomes **under certainty**, e.g. hyperbolic discounting

Most problems involve both risk AND time:

- Investment decisions
- Options
- Incentive contracts
- Annuities
- Search

Motivation for Range Utility Theory for risk and time

We dont even have a behavioral model combining loss aversion and hyperbolic discounting.

Our GOAL is to propose a general descriptive choice model for uncertain cash-flows.

Uncertain cash flows is a very general domain, and contains the important subdomains of:

- lotteries played today,
- lotteries played in the future,
- a schedule of payments under certainty,
- and a sequence of lotteries played over time, with or without serial correlation.

Motivation for Range Utility Theory for risk and time

We build on the notions of Kontek, Lewandowski (2018) and Baucells, Heukamp (2012)

KL 2018 **replace rank** principles **for range** principles.

We modify their model on three accounts:

- we introduce context dependence,
- we add reference-dependence with loss aversion.
- we relax shift and scale invariance.

Key idea 1

- A context $\mathcal{C}$ is a set of lotteries.
- It induces a range $[L, G]$
- where L the worst and G the best outcome in $\mathcal{C}$.
- Each lottery P may be evaluated:
  - context-free – the range is then $(-\infty, +\infty)$ – according to the grand range utility v
  - or context-dependent – according to $u_{[L,G]}$
- For each range, the latter is obtained as follows:

$$u_{[L,G]}(x) = \underbrace{D}_{\substack{\text{range}\\\text{effects}}} \underbrace{\left(\frac{v(x) - v(L)}{v(G) - v(L)} \right)}_{\substack{\text{Parducci}\\\text{range principle}}}, \quad x \in [L, G]. \tag{3}$$

where $D : [0, 1] \longrightarrow [0, 1]$ is continuous and strictly increasing with $D(0) = 0$, $D(1) = 1$.

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Extending the basic model

Difference to KL2018:

- The context induces the range not the lottery
- Shift and scale invariance implies:

$$u_{[L,G]}(x) = D\left(\frac{x - L}{G - L}\right), \text{ for x}\in\text{[L,G]}.$$

We relax it to get:

$$u_{[L,G]}(x) = D\left(\frac{v(x) - v(L)}{v(G) - v(L)}\right), \text{ for x}\in\text{[L,G]}.$$

where $v : X \to \mathbb{R}$ is reference-dependent with loss aversion.

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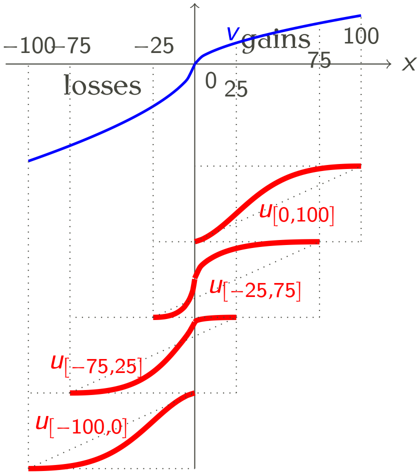
Key idea 1

Figure: The value function $v(x)$ (top) is locally distorted by range effects (bottom), yielding $u_{[L,G]}(x) = D\left(\frac{v(x)-v(L)}{v(G)-v(L)}\right)$.



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Motivation for Range Utility Theory for risk and time

BH 2012 treat time as intrinsically uncertain. They introduce probability and time-tradeoff to explain risk and time paradoxes all together.

We generalize their model from single delayed payment to uncertain cash-flows.

Key idea 2

An uncertain cash flow with given probabilities is replaced by a two stage act.

- First stage: a"horse race" is run determining the period in which the subjective termination event occurs – all the cash-flow payoffs after this period become foregone.
- Second stage: a "roulette wheel" is spun which determines the cumulative cash-flow for each termination period.

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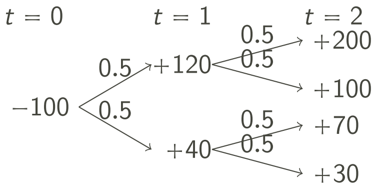
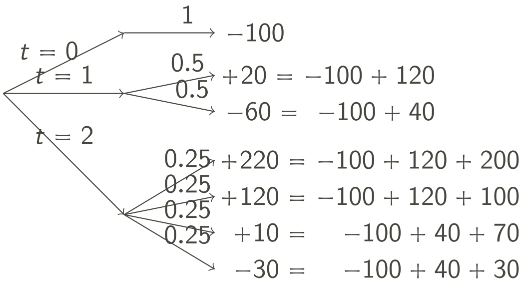
Key idea 2

An uncertain cash flow



The associated act



Structural assumption

Assumption

The decision maker is indifferent between any two cash flows that induce the same act.

Two different cash-flows induce the same act. Let $\mathbb{P}(\omega_i) = 0.125$

cash-flow 1

	0	1	2
$\omega_1 \cup \omega_2$	-100	120	200
$\omega_3 \cup \omega_4$	-100	120	100
$\omega_5 \cup \omega_6$	-100	40	70
$\omega_7 \cup \omega_8$	-100	40	30

cash-flow 2

	0	1	2
ω_1	-100	120	-10
ω_2	-100	40	70
ω_3	-100	120	-50
ω_4	-100	40	30
ω_5	-100	120	100
ω_6	-100	40	180
ω_7	-100	120	200
ω_8	-100	40	280

the AA act

	0	1	2
-100	1	0	0
-60	0	0.5	0
-30	0	0	0.25
10	0	0	0.25
20	0	0.5	0
120	0	0	0.25
220	0	0	0.25

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Range and rank principles agree for binary gambles

According to (3), the CE of a lottery $(L, G; 1-p, p)$, $L < G$, is given by

$$D\left(\frac{v(\mathrm{CE}) - v(L)}{v(G) - v(L)}\right) = (1-p)D(0) + pD(1) = p.$$

We apply D^{-1} to both sides and isolate $v(CE)$ to obtain

$$v(\mathrm{CE}) = D^{-1}(p)v(G) + (1 - D^{-1}(p))v(L). \tag{4}$$

Thus, for the case of eliciting CEs of binary lotteries, our model is preferentially equivalent to rank dependent utility.

For three or more outcomes, or binary lotteries contained on a larger context, the models diverge.

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Preference reversal

Let $(0, 120; 0.9, 0.1)$ be the \$-bet and $(0, 20; 0.2, 0.8)$ the p-bet.

Set $v(0) = 0$. When CEs are elicited each lottery is considered separately, each with its own range. The observed $CE_\$ > CE_p$ implies $v(120)D^{-1}(0.1) > v(20)D^{-1}(0.8)$.

When the two lotteries are compared *side by side*, the \$-bet dictates the range. The observed preference for the \$-bet implies $0.8D\left(v(20)/v(120)\right) > 0.1$.

The two conditions together:

$$D^{-1}\left(\frac{0.1}{0.8}\right) < \frac{v(20)}{v(120)} < \frac{D^{-1}(0.1)}{D^{-1}(0.8)},$$

which is easy to meet if D is s-shaped.

Axioms

We now state the axioms we impose on $\succsim_{\mathcal{G}} \subset \mathcal{G}^2$, $\mathcal{G} \in \mathbb{C}$.

A1 **Weak order:** Each $\succsim_{\mathcal{G}}$ is complete and transitive.

A2 **Continuity:** If $a, b, c \in \mathcal{G}$ and $a \succ_{\mathcal{G}} b \succ_{\mathcal{G}} c$ then $\alpha a + (1-\alpha)c \succ_{\mathcal{G}} b \succ_{\mathcal{G}} \beta a + (1-\beta)c$ for some $\alpha, \beta \in (0,1)$.

A3 **Independence:** If $a, b, c \in \mathcal{G}$ and $a \succ_{\mathcal{G}} b$, then $\alpha a + (1-\alpha)c \succ_{\mathcal{G}} \alpha b + (1-\alpha)c$ for all $\alpha \in (0,1]$.

A4 **Consequence Monotonicity:** If $\delta_x, \delta_y \in \mathcal{G}$ and $x > y$, then $\delta_x \succ_{\mathcal{G}} \delta_y$.

A5 **Range dependence:** If $r(\mathcal{G}) = r(\mathcal{G}')$ and $a, b \in \mathcal{G} \cap \mathcal{G}'$, then

$$a \succsim_{\mathcal{G}} b \text{ if and only if } a \succsim_{\mathcal{G}'} b.$$

Let $\succsim^*$ denote the preference relation on the grand context $\mathcal{C}^*$ and, abusing notation a little, a_t denote the constant act that offers lottery a_t in each state.

A6 **Range-principle for risk:** Any three of the following indifferences imply the fourth one:

$$\delta_x \sim p\delta_G + (1-p)\delta_L \qquad \delta_x \sim^* p'\delta_G + (1-p')\delta_L$$
$$\delta_{x'} \sim p\delta_{G'} + (1-p)\delta_{L'} \qquad \delta_{x'} \sim^* p'\delta_{G'} + (1-p')\delta_{L'}.$$

A7 **Symmetry:** If $\frac{1}{2}\delta_l + \frac{1}{2}\delta_g \sim \frac{1}{2}\delta_L + \frac{1}{2}\delta_G$ then $\frac{1}{2}\delta_l + \frac{1}{2}\delta_g \sim^* \frac{1}{2}\delta_L + \frac{1}{2}\delta_G$.

A8 **Essentiality:** For every range $[L, G]$ and $t \in \mathcal{T}$ there exist $a, b \in \mathcal{C}([L, G])$ such that $a_i = b_i$ for all $i \neq t$ and $a \succ_{\mathcal{C}([L,G])} b$.

A9 **State Monotonicity:** If $a_t \succsim_{\mathcal{C}} b_t$ for all $t \in \mathcal{T}$, then $a \succsim_{\mathcal{C}} b$.

Theorem

If preferences $(\succsim_{\mathcal{C}})_{\mathcal{C}}$, $\mathcal{C} \in \mathbb{C}$ satisfy A1–A9 if and only if there exist:

a) *a strictly increasing continuous and cardinally unique function $v : X \to \mathbb{R}$,*
b) *a unique strictly increasing, continuous and surjective function $D : [0, 1] \to [0, 1]$, such that $D(x) = 1 - D(1 - x)$, for $x \in 0, 1$*
c) *for every range $[L, G]$, a unique probability measure $\mu_{[L,G]} : \mathcal{T} \to [0, 1]$ with $\mu_{[L,G]}(t) > 0$ for each $t \in \mathcal{T}$,*

such that for any context $\mathcal{C} \in \mathbb{C}$ inducing the range $[L, G]$, the preference $\succsim_{\mathcal{C}}$ is represented by $U_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}$, as given by

$$U_{\mathcal{C}}(a) = \sum_{t=0}^{T} \mu_{[L,G]}(t) \sum_{x \in X} a_t(x) D\left(\frac{v(x) - v(L)}{v(G) - v(L)}\right), \ \forall a \in \mathcal{C}. \quad (5)$$

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Representation for risk

Theorem

If preferences $(\succsim_{\mathcal{C}})_{\mathcal{C}}$ *over constant acts,* $\mathcal{C} \in \mathbb{C}^{\text{const}}$*, satisfy axioms A1–A7 if and only if there exist functions* v *and* D *as in Theorem 4*
such that for any context $\mathcal{C} \in \mathbb{C}^{\text{const}}$ *inducing the range* $[L, G]$*, the function* $U_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}$ *that represents* $\succsim_{\mathcal{C}}$ *is given by*

$$U_{\mathcal{C}}(P) = \sum_{x \in X} P(x) D\left(\frac{v(x) - v(L)}{v(G) - v(L)}\right), \ \forall P \in \mathcal{C}. \qquad (6)$$

Subjective survival

Given the subjective probabilities of the termination events $\mu_{[L,G]} : \mathcal{T} \to [0,1]$ we define the *subjective survival function*, $S_{[L,G]} : \mathcal{T} \to [0,1]$ as follows:

$$S_{[L,G]}(t) = \sum_{i=t}^{T} \mu_{[L,G]}(i), \ \forall t \in \mathcal{T},$$

interpreted as the subjective probability of the terminating at or after t. Setting $S_{[L,G]}(T+1) = 0$, and rewriting (5):

$$U_{\mathcal{G}}(a) = \sum_{t=0}^{T} \left[S_{[L,G]}(t) - S_{[L,G]}(t+1) \right] \sum_{x \in X} a_t(x) D \left(\frac{v(x) - v(L)}{v(G) - v(L)} \right). \tag{7}$$

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Representation for cash-flows

Our preferences over acts can now be recasted as preferences over cash-flows.

$$U_{\mathcal{C}}(\tilde{X}) = \sum_{t=0}^{T} \left[S_{[L,G]}(t) - S_{[L,G]}(t+1) \right] \sum_{\omega \in \Omega} \mathbb{P}(\omega) D \left(\frac{v(\sum_{i=0}^{t} \tilde{x}_i) - v(L)}{v(G) - v(L)} \right).$$

To single out the role of discounting, we can equivalently write:

$$U_{\mathcal{C}}(\tilde{X}) = \sum_{t=0}^{T} S_{[L,G]}(t) \sum_{\omega \in \Omega} \mathbb{P}(\omega) \left[D \left(\frac{v(\sum_{i=0}^{t} \tilde{x}_i) - v(L)}{v(G) - v(L)} \right) - D \left(\frac{v(\sum_{i=0}^{t-1} \tilde{x}_i) - v(L)}{v(G) - v(L)} \right) \right]. \tag{8}$$

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Special cases

If $D(x) = x$ and $S_{[L,G]}(t) = S(t)$, then (8) becomes

$$U(\tilde{X}) = \sum_{t=0}^{T} S(t) \sum_{\omega \in \Omega} \mathbb{P}(\omega) \left[v(\sum_{i=0}^{t} \tilde{x}_i) - v(\sum_{i=0}^{t-1} \tilde{x}_i) \right]. \qquad (9)$$

For delayed lotteries, it particularizes into discounted expected utility,

$$U(\tilde{X}) = S(t) \sum_{\omega \in \Omega} \mathbb{P}(\omega) v(\tilde{x}).$$

For cash flows under certainty, the model agrees with Bell (1974) model,

$$U(\tilde{X}) = \sum_{t=0}^{T} S(t) \left[v(\sum_{i=0}^{t} x_i) - v(\sum_{i=0}^{t-1} x_i) \right].$$

Special cases

Alternatively, if v can be taken as linear (e.g., gains only, or losses only, with minor income effects), then we obtain the traditional expected discounted cash-flow model, possibly with hyperbolic discounting, given by

$$U^*_{\mathcal{C}}(\tilde{X}) = \sum_{t=0}^{T} S(t) \sum_{\omega \in \Omega} \mathbb{P}(\omega) \tilde{x}_t. \tag{10}$$

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Special cases

The CE of an uncertain CF with range $[L, G]$ solves: $D\left(\frac{v(CE)-v(L)}{v(G)-v(L)}\right) = U_{\mathcal{C}}(\tilde{X})$. Let $w(x) = D^{-1}(x)$. Then we can rewrite:

$$v(CE) = w(\pi)v(G) + (1 - w(\pi))v(L), \text{ where} \tag{11}$$

$$\pi = \sum_{t=0}^{T} S_{[L,G]}(t) \sum_{\omega\in\Omega} \mathbb{P}(\omega) \left[D\left(\frac{v(\sum_{i=0}^{t} \tilde{x}_i) - v(L)}{v(G) - v(L)}\right) - D\left(\frac{v(\sum_{i=0}^{t-1} \tilde{x}_i) - v(L)}{v(G) - v(L)}\right)\right] \tag{12}$$

For a lottery that resolves at time t, we have that

$$\pi = S_{[L,G]}(t) \sum_{\omega\in\Omega} \mathbb{P}(\omega) D\left(\frac{v(x) - v(L)}{v(G) - v(L)}\right). \tag{13}$$

For a lottery that resolves now,

$$\pi = \sum_{\omega\in\Omega} \mathbb{P}(\omega) D\left(\frac{v(x) - v(L)}{v(G) - v(L)}\right). \tag{14}$$

And for a binary lottery $(L, G; p, 1 - p)$, we have that $\pi = p$.

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We provide a novel way to generalize binary rank-dependent utility, which is at the intersection of numerous choice model.

If v is linear, then (14) becomes the range-dependent utility (Kontek, Lewandowski, 2018). Thus, (14) extends range-dependent utility to losses, (13) includes delay, and (12) adds multiple cash flows.

For a delayed binary prospect with $L = 0$, Baucells, Heukamp (2012) provide axiomatic foundations for the discounted probability approach $v(CE) = w(e^{-r_G(t)}P)v(G)$. Our model yields $v(CE) = w(S_{[0,G]}(t)P)v(G)$, and can be seen as a generalization of the discounted probability approach not only to delayed lotteries with multiple outcomes, but also to uncertain cash flows, possibly with context effects.