SummerLab
Games & Decisions

Part I: Range effects for decisions under risk and time delay

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Outline of this part

1. Expected Utility Theory
2. Expected Utility paradoxes
3. Behavioral critique and Prospect Theory
4. Range-dependent utility model for risk
5. Paradoxes for risk and time
6. Range utility theory for risk and time

Setup

X - the set of consequences;

$\Delta(X)$ - the set of lotteries denoted by P, Q, R, i.e. finite-support probability distributions over X

$\succsim \subset \Delta(X) \times \Delta(X)$ - preference relation, i.e. binary relation over lotteries

We will say that a function $U : \Delta(X) \to \mathbb{R}$ represents a preference relation $\succsim$ if $P \succsim Q \iff U(P) \geqslant U(Q)$.

The symmetric and asymmetric parts of $\succsim$, i.e. $\sim$ and $\succ$, respectively, are defined the standard way.

Given two lotteries P, Q and a number $\alpha \in [0, 1]$, $\alpha P + (1 - \alpha)Q$ is also a lottery, called a mixture of P and Q.

Expected Utility axioms

$\succsim$ satisfies three axioms:

1. Weak order ($\succsim$ is complete and transitive)
2. Archimedean: if $P \succ Q \succ R$, then there exist numbers $\alpha, \beta \in (0, 1)$, such that $\alpha P + (1 - \alpha)R \succ Q \succ \beta P + (1 - \beta)R$.
3. Independence: If $P \succ Q$, then $\alpha P + (1 - \alpha)R \succ \alpha Q + (1 - \alpha)R$, for all $\alpha \in (0, 1)$ and R.

Theorem (von Neumann, Morgenstern, 1944)

A preference relation $\succ \subseteq \Delta(X) \times \Delta(X)$ *satisfies Axioms 1-3*

if and only if

(Existence:) *There is a function* $u : X \to \mathrm{R}$ *such that the function* $U : \Delta(x) \to \mathbb{R}$ *defined by* $U(P) = \sum_{x \in \text{supp}(P)} u(x)P(x)$ *for each* $P \in \Delta(X)$ *represents* $\succsim$*.*

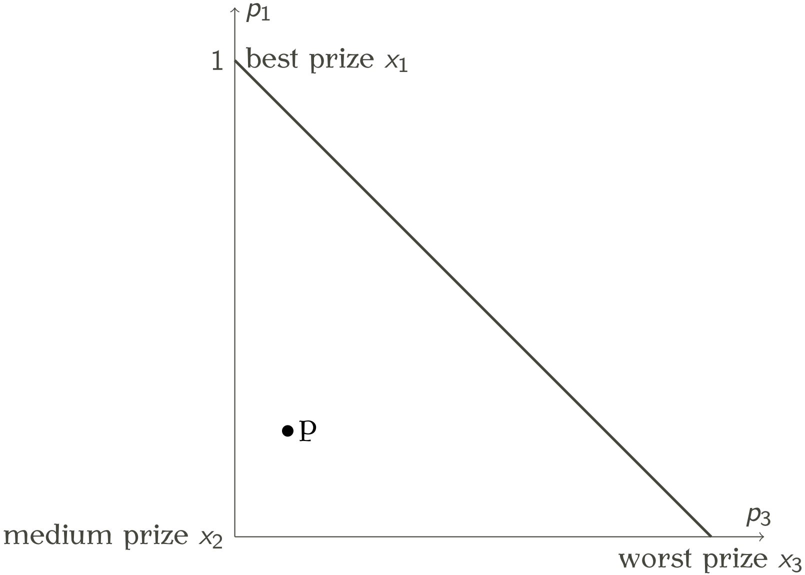
(Uniqueness:) *Moreover, if* u *provides a representation of* $\succsim$ *in this sense, then* v *does as well if and only if there exist* $a, b \in \mathrm{R}$, $a > 0$, *such that* $v(x) = au(x) + b$ *for all* $x \in X$.

Crucial implications of the axioms

Two implications of the axioms:

1. $\sim$ Betweenness: if $P \sim Q$ then $P \sim \alpha P + (1 - \alpha) Q \sim Q$.
2. $\sim$ Independence: if $P \sim Q$ then
   $\alpha P + (1 - \alpha) R \sim \alpha Q + (1 - \alpha) R$.

The Marschack-Machina triangle



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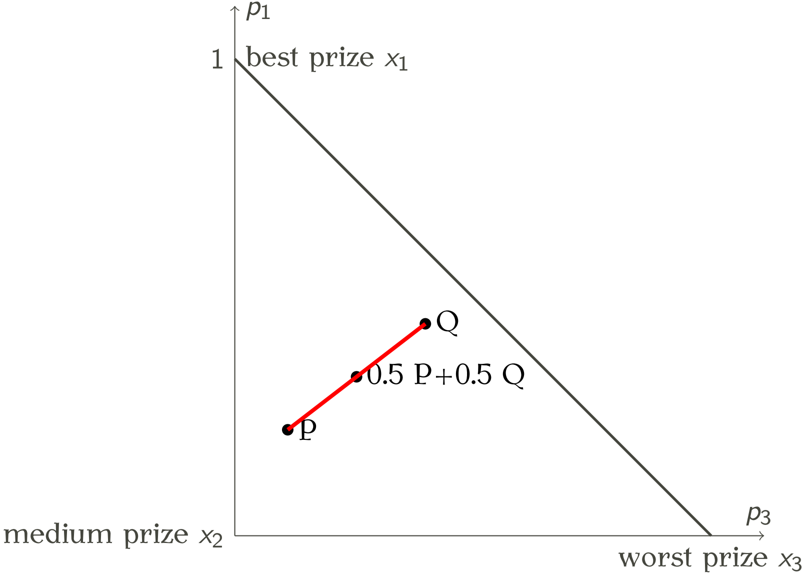

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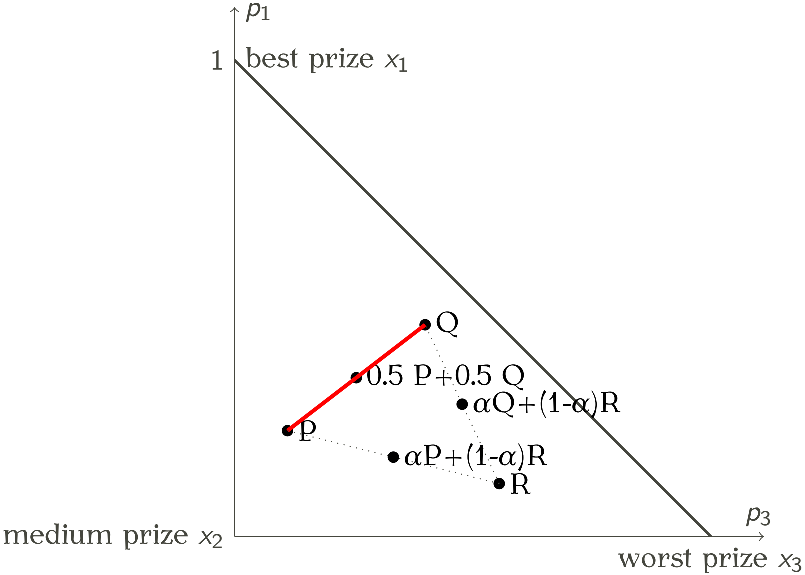

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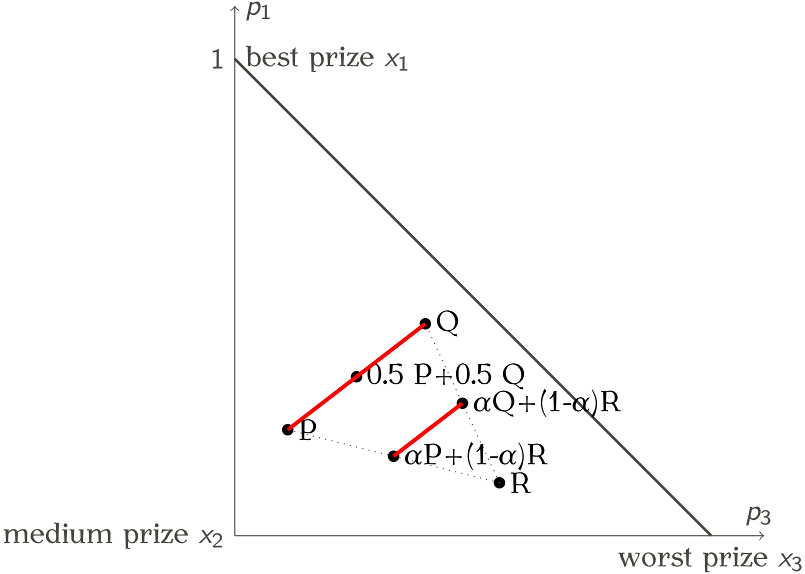


Hidden assumption - Independence of states

An act is a finite-valued mapping $f : S \to X$, where S is the set of states of nature (mutually exclusive and exhaustive).

Let π be a well defined probability measure defined on all subsets of S (we assume the full algebra).

Then the probability distribution of an act f is a mapping $P_f : X \to [0, 1]$ such that $P_f(x) = \int_{\{s \in S : f(s) = x\}} f(s)$.

Expected Utility Theory implicitly assumes that preferences over acts are reducible to preferences over their probability distributions, i.e. they are **independent of the state space**.

We will henceforth exchangeably write $f \succsim g$ meaning that $P_f \succsim P_g$.

In order to apply Expected Utility theory, one needs to adopt an **interpretation**.

The leading interpretation for many years was that of **consequentialism** (Rubinstein, 2012).

It states that

1. there is a single preferences relation $\succsim$ over lotteries defined on wealth levels (X is the set of wealth levels)
2. preferences over lotteries defined on wealth changes $\succsim_W$, where W is a given wealth level, are derived from $\succsim$ by:
$$f \succsim_W g \iff W + f \succsim W + g.$$

Expected Utility paradoxes

Starting from the fifties, numerous evidence accumulated showing that people violate Expected Utility. These violations include:

1. Violations of independence
2. Violations of procedural invariance
3. Violations of description invariance
4. Matching the observed risk attitudes,

Consequences in thousand dollars.

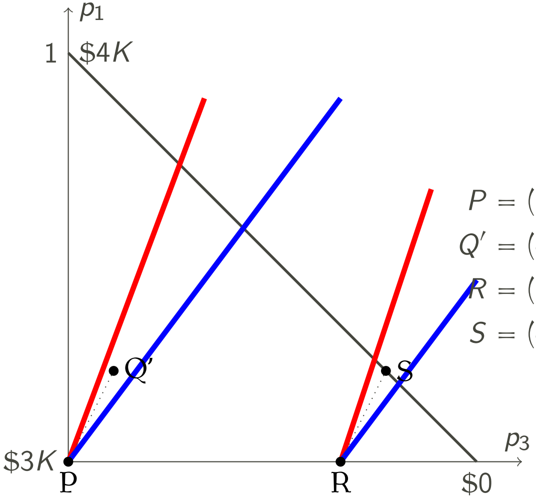
Common consequence effect (Allais, 1954)

$$P = (3, 1) \succ (4, 0.2; 3, 0.75; 0, 0.05) = Q'$$
$$R = (3, 0.25; 0, 0.75) \prec (4, 0.2; 0, 0.8) = S$$

Common ratio effect

$$P = (3, 1) \succ (4, 0.8; 0, 0.2) = Q$$
$$R = (3, 0.25; 0, 0.75) \prec (4, 0.2; 0, 0.8) = S$$

Common consequence effect in the MM triangle



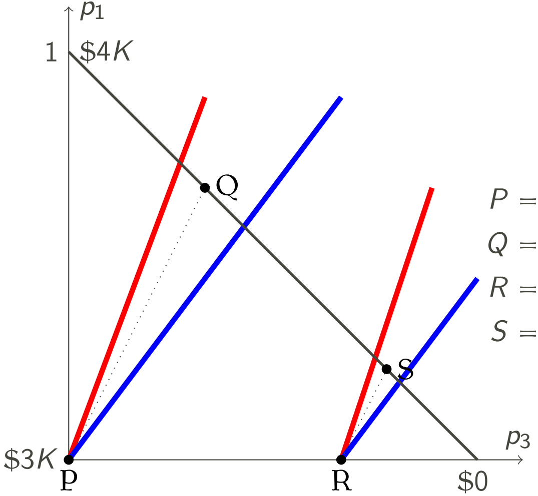
$P = (3, 1)$

$Q' = (4, 0.2; 3, 0.75; 0, 0.05)$

$R = (3, 0.25; 0, 0.75)$

$S = (4, 0.2; 0, 0.8)$

Common ratio effect in the MM triangle



$P = (3, 1)$

$Q = (4, 0.8; 0, 0.2)$

$R = (3, 0.25; 0, 0.75)$

$S = (4, 0.2; 0, 0.8)$

Violations of procedural invariance: **WTA-WTP disparity**

WTA is the smallest price I am willing to accept for the lottery I own.

WTP is the highest price I am willing to pay for the lottery

In the Expected Utility of wealth model WTA and WTP may differ only due to wealth effects

Starting form Knetsch, Sinden (1984), there were numerous studies showing that for lotteries WTA often exceeds WTP by more that 100%. (more on that later)

Violations of procedural invariance: **Preference reversal**

For a lottery P, the certainty equivalent is the sure amount, such that the DM is indifferent between P and this amount.

There are two lotteries: the \$-bet $(\$120, 0.1; \$0, 0.9)$ and the P-bet $(\$20, 0.8; \$0, 0.2)$.

Preference reversal occurs when the DM prefers the P-bet in a direct choice but assigns higher Certainty Equivalent to the \$-bet.

This phenomenon was first analyzed by Lichtenstein, Slovic (1971) and Grether, Plott (1979).

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Violations of description invariance: **Narrow framing**

First introduced by Kahneman, Tversky (1981). This effect occurs if depending on the description of different alternatives people change their decision.

Suppose a community is preparing for the outbreak of an Asian disease which is expected to kill 600 people. You may choose between the following two programs expressed in terms of the number of lives saved:

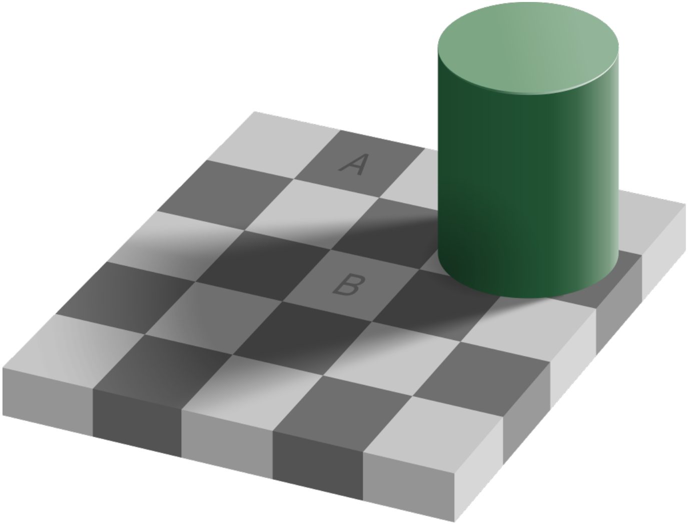
$$200 \text{ vs. } (600, 1/3; 0, 2/3)$$

Most people choose the sure option. If the same problem is expressed in terms of lives lost:

$$-400 \text{ vs. } (0, 1/3; -600, 2/3)$$

then most people reverse their choice and prefer the risky option.

Framing



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Framing

In addition to what you own, you are given \$2,000. Which option do you prefer:

- A $-\$500$ for sure
- B' $(-\$1000, 0.5; \$0, 0.5)$

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In addition to what you own, you are given \$1,000. Which option do you prefer:

- A' $\$500$ for sure
- B' $(\$1000, 0.5; \$0, 0.5)$

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Framing

In addition to what you own, you are given \$2,000. Which option do you prefer:

A $-\$500$ for sure

B' $(-\$1000, 0.5; \$0, 0.5)$

In addition to what you own, you are given \$1,000. Which option do you prefer:

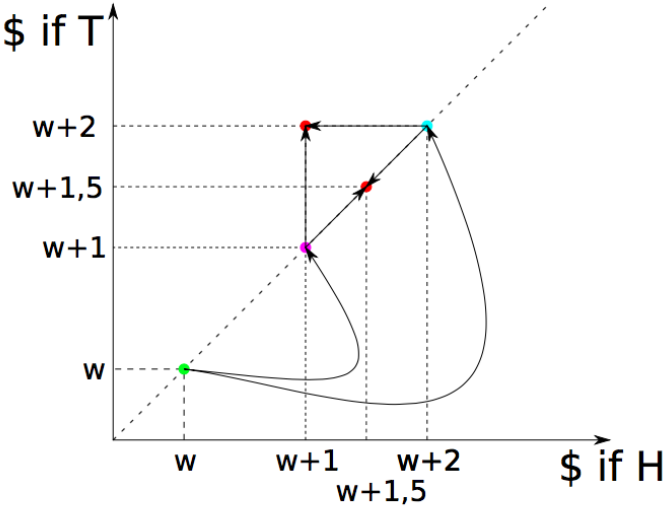
A' \$500 for sure

B' $(\$1000, 0.5; \$0, 0.5)$

These two situations are payoff-equivalent:

- gamble A and A': $(w + 1.5, w + 1.5)$,
- gamble B and B': $(w + 2, w + 1)$

Yet people make different choices.



The slide taken from Tomasz Strzalecki lecture notes for *Psychology and Economics*.

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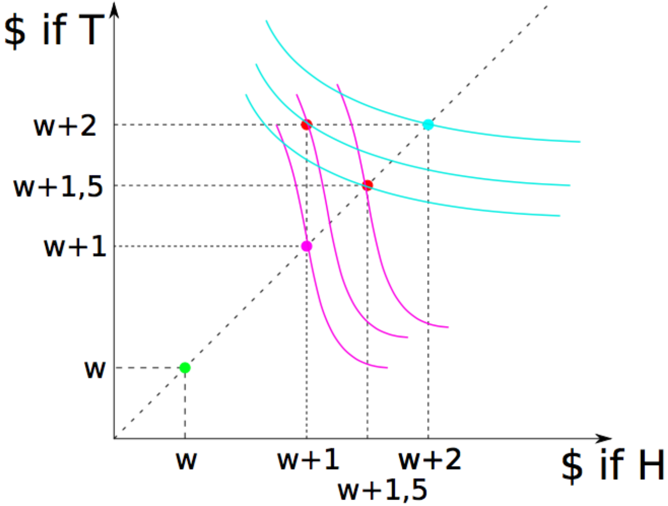
Framing



The slide taken from Tomasz Strzalecki lecture notes for *Psychology and Economics.*

Tomek Strzalecki (Harvard University) who axiomatized the robust control model of Hansen, Sargent in spite of a firm belief it cannot be done.

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Problem

Problem 1:

Decision (i) Choose between:

A. A sure gain of \$240
B. 25% chance to gain \$1,000,
   75% chance to lose nothing

Decision (ii) Choose between:

C. A sure loss of \$750
D. 75% chance to lose \$1,000,
   25% chance to lose nothing.

Problem

Problem 1:

Decision (i) Choose between:

A. A sure gain of $240
B. 25% chance to gain $1,000, 75% chance to lose nothing

Decision (ii) Choose between:

C. A sure loss of $750
D. 75% chance to lose $1,000, 25% chance to lose nothing.

Did you choose A and D?

Problem 2:

Choose between:

E. 25% chance to win $240 and 75% chance to lose $760
F. 25% chance to win $250 and 75% chance to lose $750.

The status quo bias and the endowment effect

The status quo bias: First introduced by Samuelson, Zeckhauser (1988). It occurs if people prefer things to stay the same as they used to be by doing nothing or by sticking with the decision made previously.

Endowment effect: First demonstrated by Knetsch, Sinden (1984), it occurs when people overvalue a good that they own, regardless of its objective market value. One famous example concerning sure coffee mugs was given by Kahneman *et al.* (1990).

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Coffee mugs experiment

Kahnemann, Knetsch, Thaler (1990)

- half of the class is randomly given mugs
- those with mugs are asked to value them
- those without mugs are asked to bid for them
- the median Willingness to Pay (WTP) is \$2.50 and the median Willingness to Accept (WTA) is \$5.25
- the market clears
- if nobody had loss aversion, this market would result in trade equal to half of the quantity of the mugs
- yet, in reality few mugs change hands (about 10%)

Reference dependence: first proposed by Markowitz (1952); people care about changes rather than levels of wealth; outcomes are evaluated relative to some reference point (often the status quo)

Loss aversion: means that losses loom larger than gains; people dislike gambles of the form $(x, 0.5; -x, 0.5)$, where $x \neq 0$. Furthermore, if $0 < |x| < |y|$ then people usually express the following preference:
$(x, 0.5; -x, 0.5) \succ (y, 0.5; -y, 0.5)$.

Rabin paradox

Rabin (2000) - calibration theorem

If an EU agent rejects lottery $(\$110, 0.5; -\$100, 0.5)$ at any initial wealth level

Then s/he will also reject the lottery $(+\$\infty, 0.5; -\$1000, 0.5)$.

The reflection effect

TABLE I

PREFERENCES BETWEEN POSITIVE AND NEGATIVE PROSPECTS

Positive prospects				Negative prospects			
Problem 3:	(4,000, .80)	<	(3,000).	Problem 3′:	(−4,000, .80)	>	(−3,000).
$N = 95$	[20]		[80]*	$N = 95$	[92]*		[8]
Problem 4:	(4,000, .20)	>	(3,000, .25).	Problem 4′:	(−4,000, .20)	<	(−3,000, .25).
$N = 95$	[65]*		[35]	$N = 95$	[42]		[58]
Problem 7:	(3,000, .90)	>	(6,000, .45).	Problem 7′:	(−3,000, .90)	<	(−6,000, .45).
$N = 66$	[86]*		[14]	$N = 66$	[8]		[92]*
Problem 8:	(3,000, .002)	<	(6,000, .001).	Problem 8′:	(−3,000, .002)	>	(−6,000, .001).
$N = 66$	[27]		[73]*	$N = 66$	[70]*		[30]

The table taken from Kahnemann, Tversky, 1979

The reflection effect, four-fold pattern of risk attitudes and coexistence of insurance and gambling

The reflection effect means that the preference between loss prospects is the mirror image of the preferences between gain prospects.

The reflection effect together with the certainty/possibility effect implies the so called **four-fold pattern**:

	small probability	large probability
gain	risk seeking	risk aversion
loss	risk aversion	risk seeking

This pattern also accommodates the phenomenon of coexistence of insurance and gambling.

Other violations

1. **Violations of monotonicity**
   - People choose stochastically dominated lotteries,
   - Let F_P denote the CDF of a lottery P. We say that P dominates Q if $F_P(x) \leqslant F_Q(x)$ for all $x \in \mathbb{R}$ with strict inequality for at least one x.
2. **Violations of transitivity**
   - It occurs if people choose P over Q, Q over R and R over P.
3. **event-splitting effects**
   - Starmer, Sugden (1993) demonstrated that when an event that gives a given outcome is split into two sub-events, there is a tendency for that outcome to carry more weight even though its total probability is unchanged.
4. Etc.

Prospect Theory

Motivated by the EU paradoxes, Kahnemann, Tversky (1979) and then Tversky, Kahnemann (1992) proposed **Prospect Theory**. Its crucial elements are:

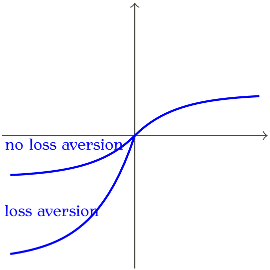
- **Reference dependence** with loss aversion
- **Probability weighting**

Reference dependence replaces consequentialism. It is a different interpretation.

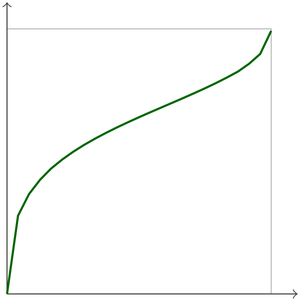
Probability weighting challenges Independence, the key axiom of Expected Utility Theory.

Value and weighting functions

Value function



Probability weighting function



Is Expected utility an Ex-hypothesis?

Rabin, Thaler (2001) declared Expected Utility an ex-hypothesis or a dead parrot alluding to the famous sketch from Monthy Pythons Flying Circus.

We feel much like the customer in the pet shop, beating away a ***dead parrot****.*

EU theory vs model

What Rabin, Thaler (2001) criticize is in fact not Expected Utility Theory, but the model of Expected Utility of lifetime-wealth.

One should distinguish between theory and model:

1. **theory**: abstract, mathematical, axioms
2. **model**: theory plus interpretation

The model of Expected Utility of lifetime-wealth combines EU theory and consequentialism.

Abandoning consequentialism and replacing it with other interpretations while keeping Expected Utility removes many EU paradoxes, for example the Rabin paradox (Cox, Sadiraj, 2006, Palacios-Huerta, Serrano, 2006).

Different models based on Expected Utility

Different interpretations consistent with Expected Utility Theory:

1. **Expected Utility of gambling wealth**: EU theory and mental accounting (Lewandowski, 2014, Foster, Hart, 2009)
2. **Reference-Dependent Expected Utility** (Sugden, 2003, Schneider, Day, 2016, Lewandowski, 2019)
3. **Range-dependent utility** (Kontek, Lewandowski, 2018)

Following Palacios-Huerta, Serrano (2006) we may cite Mark Twain and say:

The reports of my death were an exageration.

Range-dependent utility - outline

1. Range effects (Parducci, 1964)
2. Based on Parducci, we propose **range-dependent utility (RDU)** for risk – as general theory.
3. Based on Tversky, Kahnemann (1992) experimental data we propose the **decision utility model (DU)** - special case of RDU used for prediction:
4. Important properties of the model: Monotonicity wrt FOSD and continuity

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Allen Parducci (Pacific Palisades) who coinvented windsurfing and proposed one of the most famous theory of psychophysical judgment

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Range-frequency theory

Range-Frequency Theory was proposed by Parducci (1965)

Psychophysical judgment is a compromise between two principles:

1. **the range principle** R
   - subjects locate each stimulus relative to the subjective end values.
   - Let $s_1, \ldots, s_i, \ldots, s_N$ be the stimulus values in the context of stimuli affecting the judgment of s_i arranged in the increasing order.

$$R_i = \frac{s_i - s_{\min}}{s_{\max} - s_{\min}},$$

2. **the frequency principle** F
   - differences in response are proportional to differences in stimulus rank
   - $1, \ldots, i, \ldots, N$ denote the stimuli ranks:

$$F_i = \frac{i-1}{N-1},$$

Range effects

We focus on range effects. Example:

1. Tropical island, the temperature always in the 80s, the natives complain of:
   - the heat when the temp is 88,
   - the cold when the temp is 82.

   For us, on the contrary, such differences are hardly noticeable.

Other examples:

1. Expensive dish in the restaurant menu.
2. Eye-adaptation process. Two theories:
   - Adapation-level theory (Helson, 1963)
   - Range-frequency theory (Parducci, 1964)

Tversky, Kahnemann (1992) data

Data used to motivate Cumulative Prospect Theory:

No	xl	xu	p	CE
1	0	50	0,10	9,0
2	0	50	0,50	21,0
3	0	50	0,90	37,0
4	0	100	0,05	14,0
5	0	100	0,25	25,0
6	0	100	0,50	36,0
7	0	100	0,75	52,0
8	0	100	0,95	78,0
9	0	200	0,01	10,0
10	0	200	0,10	20,0
11	0	200	0,50	76,0
12	0	200	0,90	131,0
13	0	200	0,99	188,0
14	0	400	0,01	12,0
15	0	400	0,99	377,0
16	50	100	0,10	59,0
17	50	100	0,50	71,0
18	50	100	0,90	83,0
19	50	150	0,05	64,0
20	50	150	0,25	72,5
21	50	150	0,50	86,0
22	50	150	0,75	102,0
23	50	150	0,95	128,0
24	100	200	0,05	118,0
25	100	200	0,25	130,0
26	100	200	0,50	141,0
27	100	200	0,75	162,0
28	100	200	0,95	178,0

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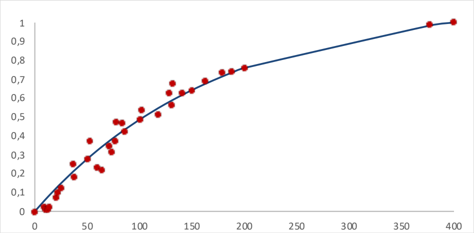
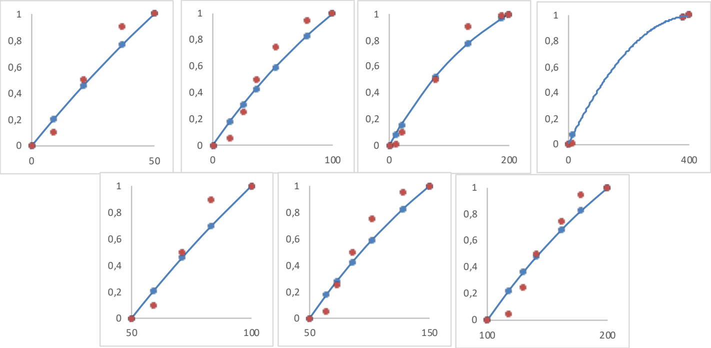
Expected Utility fits poorly





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Assignment

1. Fit the data with the Expected Utility model:
   a) Choose arbitrary utility values for the outcomes 50, 100, 150, 200, and $u(0) = 0$, $u(400) = 1$.
   b) Assume that the vNM utility function has the CARA form: $u(x) = \frac{1-e^{-\alpha x}}{1-e^{-\alpha \mathbf{400}}}$, for $x \in [0, 400]$.
   c) Invert the function and apply it to the utility values to obtain the theoretical CE values.
   d) Minimize the sum of squared deviation of the real CE values from the theoretical ones by adjusting the vNM utility function parameter α and the utility values that you were asked to arbitrarily choose at the beginning.
2. To see that the fit is poor, you need to portion the data into lotteries having the same range and then superimpose the function that you have found.

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TK (1992) data

No	xl	xu	p	CE
1	0	50	0.10	9.0
2	0	50	0.50	21.0
3	0	50	0.90	37.0
4	0	100	0.05	14.0
5	0	100	0.25	25.0
6	0	100	0.50	36.0
7	0	100	0.75	52.0
8	0	100	0.95	78.0
9	0	200	0.01	10.0
10	0	200	0.10	20.0
11	0	200	0.50	76.0
12	0	200	0.90	131.0
13	0	200	0.99	188.0
14	0	400	0.01	12.0
15	0	400	0.99	377.0
16	50	100	0.10	59.0
17	50	100	0.50	71.0
18	50	100	0.90	83.0
19	50	150	0.05	64.0
20	50	150	0.25	72.5
21	50	150	0.50	86.0
22	50	150	0.75	102.0
23	50	150	0.95	128.0
24	100	200	0.05	118.0
25	100	200	0.25	130.0
26	100	200	0.50	141.0
27	100	200	0.75	162.0
28	100	200	0.95	178.0

xl	xu	p	CE
0	50	0.00	0
0	50	0.10	9.0
0	50	0.50	21.0
0	50	0.90	37.0
0	50	1.00	50.0

⋮

xl	xu	p	CE
100	200	0.00	100.0
100	200	0.05	118.0
100	200	0.25	130.0
100	200	0.50	141.0
100	200	0.75	162.0
100	200	0.95	178.0
100	200	1.00	200.0

- Fix the range $[L, G]$
- Assign $u_{[L,G]}(L) = 0$ and $u_{[L,G]}(G) = 1$
- Construct utility: $u_{[L,G]}(CE) = p$

- Here we fit a nonlinear, strictly increasing and surjective function $u_{[L,G]} : [L, G] \to [0, 1]$.

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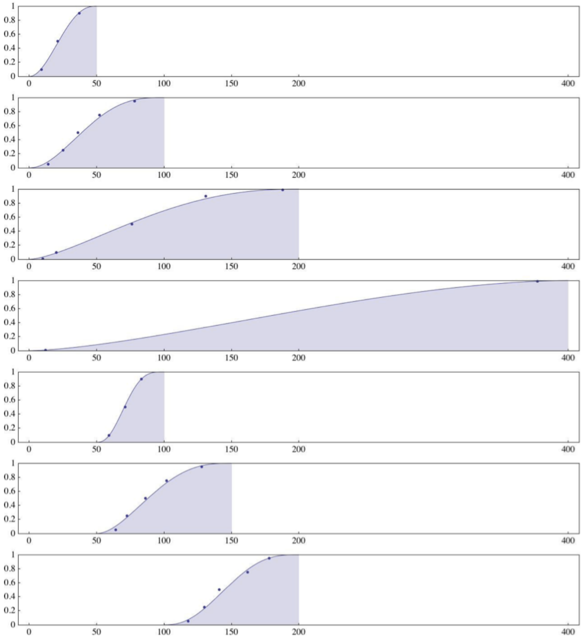
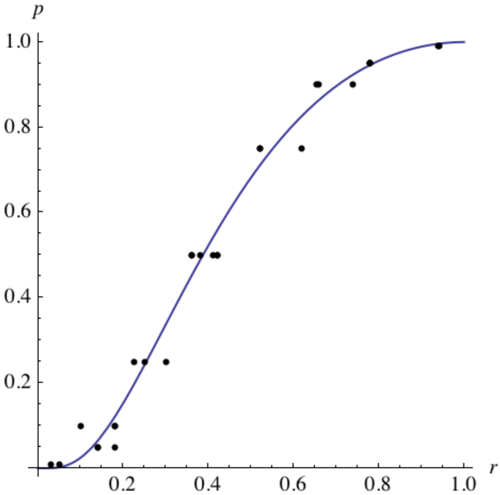
Fitting range-dependent utility functions



- The fit is very good
- But requires a separate utility for each range
- Observe, however, that the graphs differ mostly in shift and scale of the consequences.

Fitting the decision utility function

- Normalize all lottery ranges and consequences into a common interval $[0,1]$
- Define a single function $D : [0,1] \to [0,1]$, called the **decision utility function**
- The fit is still very good,
- but now we have only one utility.



Setup

- X – set of monetary alternatives
- Δ – set of finite-support **lotteries**
  - Δ^d – set of degenerate lotteries
  - A degenerate lottery δ_x assigns probability 1 to a single consequence x.
- **the range of lottery** P Conv(supp$P$) is a proper real interval $[L, G]$.
- $\Delta^c_{[L,G]}$ – set of lotteries that are **comparable in the range** $[L, G]$ is the union of two sets:
  - $\Delta_{[L,G]}$ – set of lotteries with range equal to $[L, G]$
  - $\Delta^d_{[L,G]}$ – set of degenerate lotteries with support in $[L, G]$

Axioms

A "range-dependent" preference relation $\succsim \subset \Delta \times \Delta$ satisfies the following axioms:

Axiom (1)

Weak Order: *$\succsim$ is complete and transitive.*

Axiom (2)

Within-Range Continuity: *For any interval $[L, G] \subset X$, $L < G$ and for every $Q \in \Delta^c_{[L,G]}$ the following holds:*

$$\delta_G \succ Q \succ \delta_L \Longrightarrow$$
$$\exists \alpha, \beta \in (0, 1) : \alpha\delta_G + (1 - \alpha)\delta_L \succ Q \succ \beta\delta_G + (1 - \beta)\delta_L.$$

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Axiom (3)

Within-Range Independence: *For any interval* $[L, G] \subset X$, $L < G$, *for every* $P, Q, R \in \Delta$, *such that* $\alpha P + (1-\alpha)R, \alpha Q + (1-\alpha)R \in \Delta^c_{[L,G]}$, *for all* $\alpha \in (0, 1]$ *the following holds:*

$$P \succsim Q \iff \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R, \ \forall \alpha \in [0, 1].$$

Axiom (4)

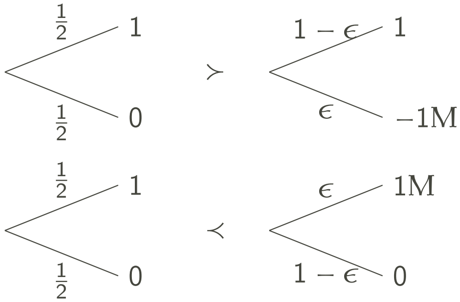
Monotonicity: *For all* $x, y \in X$ *the following holds:*

$$x > y \iff \delta_x \succ \delta_y$$

Discussion on axioms

Continuity and Independence are weakened.

Within-Range Continuity: allows violations of continuity when lottery ranges differ

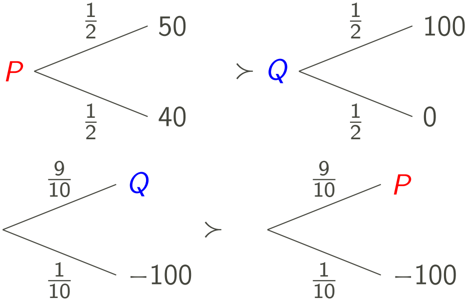


for all $\epsilon > 0$

for all $\epsilon > 0$

Discussion on axioms

Within-Range Independence: allows violations of independence when lottery ranges differ.



Range-dependent utility representation

Theorem (**General theory**)

A preference relation $\succsim \subset \Delta \times \Delta$ *satisfies axioms A1–A4*

if and only if

for every interval $[L, G] \subset X, L < G$ *there exists a unique strictly increasing and surjective function* $u_{[L,G]} : [L, G] \to [0, 1]$, *such that for every pair of lotteries* $P, Q \in \Delta$ *the following holds:*

$$P \succsim Q \iff \mathrm{CE}(P) \geqslant \mathrm{CE}(Q), \qquad (1)$$

where the certainty equivalent is defined as:

a) $\boxed{\mathrm{CE}(P) = u^{-1}_{\mathrm{Rng}(P)} \left[\sum_{x \in X} P(x) u_{\mathrm{Rng}(P)}(x)\right]}$ *for any* $P \in \Delta \setminus \Delta^d$,

b) $\mathrm{CE}(\delta_x) = x, x \in X$ *for any* $\delta_x \in \Delta^d$.

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Intuition

- The same consequence might be assigned two different utility values depending which lottery (with its range) it supports.
- CE values, instead of utility values, are used to represent choices between lotteries having different ranges.

The intersection of range-dependent utility and EU

1. **The case of a universal range**: In real life there always exists a tiny chance to die at once or to find a billion dollars on the street.
   - Broad framers always use a universal range (and hence are rational); narrow framers on the other hand exhibit EU paradoxes
2. **The case of consequentialism**: Let u be a utility over lifetime wealth levels. We accommodate the EU model of lifetime wealth by taking: $u_{[L,G]}(x) = \frac{u(W+x)-u(W+L)}{u(W+G)-u(W+L)}$, $\forall x \in [L, G]$, for each interval $[L, G]$.

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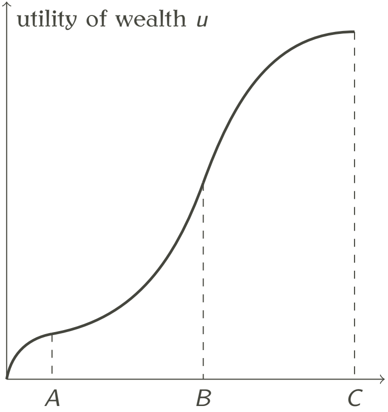
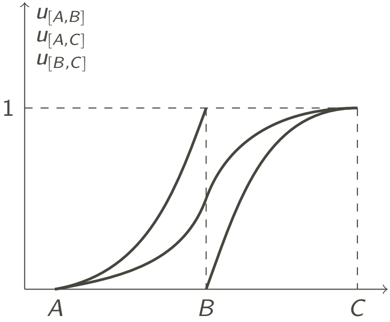
The case of consequentialism





Additional axiom: Shift and scale invariance

Definition

For a lottery $P \in \Delta$, $P : X \to [0,1]$ *define its* α,β-***transformation*** $P_{\alpha,\beta} \in \Delta$, $P_{\alpha,\beta} : X \to [0,1]$, *such that* $P(x) = P_{\alpha,\beta}(\alpha x + \beta)$, *where* $\alpha,\beta \in \mathbb{R}$, $\alpha > 0$, $x \in X$ *and* $\alpha x + \beta \in X$, *for all* $x \in \operatorname{supp}(P)$.

Axiom (5)

Scale and Shift invariance: *Let* $P, Q \in \Delta^c_{[L,G]}$ *for some* $[L,G] \subset X$, $L < G$. *Then the following holds:*

$$P \succsim Q \iff P_{\alpha,\beta} \succsim Q_{\alpha,\beta}$$

for any $\alpha > 0, \beta \in \mathbb{R} : \; P_{\alpha,\beta}, Q_{\alpha,\beta} \in \Delta^c_{\alpha L+\beta,\alpha G+\beta}$.

In what follows it is assumed that $[0,1] \subset X$.

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Theorem (**Model for prediction**)

A preference relation $\succsim \subset \Delta \times \Delta$ *satisfies axioms A1–A5*

if and only if

there exists a unique strictly increasing and surjective function $D : [0,1] \to [0,1]$*, such that for every pair of lotteries* $P, Q \in \Delta$ *the following holds:*

$$P \succsim Q \iff \mathrm{CE}(P) \geqslant \mathrm{CE}(Q), \tag{2}$$

where the certainty equivalent is defined as:

a) $\boxed{\mathrm{CE}(P) = L + (G - L)D^{-1}\left[\sum_{x \in X} P(x) D\left(\frac{x-L}{G-L}\right)\right]}$*, for any* $P \in \Delta \setminus \Delta^d$*, where* $L = \min(\mathrm{Rng}(R))$, $G = \max(\mathrm{Rng}(R))$,

b) $\mathrm{CE}(\delta_x) = x, x \in X$ *for any* $\delta_x \in \Delta^d$.

Discussion on the axiom

1. The family $(u_{[L,G]})$ is induced from a single decision utility function D by taking:

$$u_{[L,G]}(x) := D\left(\frac{x-L}{G-L}\right), \ \forall x \in [L,G].$$

2. Due to axiom (5) the model exhibits **Constant Risk Aversion** of Safra and Segal (1998)
   - The model intersects EU in the case of risk neutrality
     - under EU: shift invariance ≡ CARA, scale invariance ≡ CRRA, shift and scale invariance ≡ linear utility

Observational equivalence btw. Decision Utility and Dual Theory (rank dependence)

Consider a binary lottery payoff $(L, 1-p; G, p)$

$$\textbf{Decision utility: } \mathrm{CE}(x) = L + (G-L)D^{-1}(p),$$

$$\textbf{Dual Theory: } \mathrm{CE}(x) = L + (G-L)w(p).$$

- The same predictions iff $D^{-1} = w$.
- Evidence for binary lottery provides equal support for probability weighting and range dependence.
- For more than 2 outcomes the models can be discriminated.

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