Foster-Hart measure fo riskiness and buying/selling price for a lottery

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Abstract

This paper establishes a number of functional relationships between the riskiness measure of Foster-Hart and its extension and buying/selling price of a lottery. The results allow comparison of riskiness measures for lotteries that either have non-positive expectation or do not take negative values.

Keywords: operational measure of riskiness, buying, selling price for a lottery

JEL Classification Numbers: D81, D03, C91

1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space, fixed for the rest of the paper. Let S be some real interval. A random variable defined on this space, $X : \Omega \to S$, is called a lottery. The space of all lotteries is denoted by $\mathcal{X}$. The probability distribution of $X \in \mathcal{X}$ is a mapping $P_X : S \to [0,1]$ such that

$$P_X(x) = \sum_{\omega \in \Omega : X(\omega) = x} \mathbb{P}(\omega), \text{ for all } x \in \text{supp}(X),$$

where $\text{supp}(X) = \{x \in S : X(\omega) = x, \text{ for some } \omega : \mathbb{P}(\omega) > 0\}$. Note that lotteries have finite support because the underlying probability space is finite. We focus on Expected Utility maximizers. For them the only part of a lottery relevant for making decisions is its probability distribution. Hence, for simplicity, we will refer to both $X \in \mathcal{X}$ as well as its probability distribution P_X as to a lottery.

Expectations are taken over the relevant probability distributions. A typical lottery will be denoted as $\langle x_1, ..., x_n; p_1, ..., p_n \rangle$, where $x_i \in S$ are outcomes and $p_i \in [0,1]$ are the corresponding probabilities. It is assumed w.l.o.g. that its consequences are ordered $x_1 < x_2 < ... < x_n$. We will also denote $\min(X)$ as the minimal element in the support of X and $\mathbb{E}(X)$ as the expected value of X. Denote by $\mathcal{X}_R$ as the set of lotteries with positive expectation and possible losses. Formally:

$$\mathcal{X}_R = \{X \in \mathcal{X} : \ \mathbb{E}(X) > 0, \ \min(X) < 0\}.$$

Consequences in S are assumed monetary and interpreted as wealth positions.

Assumption 1.1. *Preferences over lotteries obey expected utility axioms. The vNM utility function $U : S \to \mathbb{R}$ is strictly increasing, strictly concave and continuous.*

Remark 1.1. *Strict monotonicity means that more sure money is better than less. Strict concavity is equivalent to risk aversion: for any nondegenerate lottery, the decision maker prefers to receive the expected monetary value of a lottery to a lottery itself.*

Remark 1.2. *Continuity is a technical requirement. However, note that any strictly increasing concave function $U : S \to \mathbb{R}$ is continuous on every interval $(a, b] \subset S$ so the condition of continuity is only relevant if S contains its lower boundary points.*

Remark 1.3. *Any utility function satisfying Assumption 1.1 decreases at least linearly as* $x \to \inf S$*. Hence* U *cannot be bounded from below unless* $\inf S > -\infty$*.*

We will add more assumption on U as we go along. Specifically, if U is twice continuously differentiable, the Arrow-Pratt measure of (absolute) risk aversion is a function $\mathrm{ARA}(x) = \frac{-U''(x)}{U'(x)}$, defined on the $\mathrm{int}S$. The (relative) risk aversion is a function $\mathrm{RRA}(x) = x\mathrm{ARA}(x)$.

Definition 1.1. *A twice continuously differentiable utility function* $U : (0, \infty) \to \mathbb{R}$ *satisfying Assumption 1.1 belongs to the DARA class if ARA is a decreasing function. It belongs to the CRRA class if RRA is a constant function. In the latter case, up to linear transformation* U *takes the form:*

$$U_\alpha(x) = \begin{cases} \frac{x^{1-\alpha}-1}{1-\alpha}, & 0 < \alpha < 1, \quad x \geq 0 \\ \log x, & \alpha = 1, \quad x > 0 \\ \frac{x^{1-\alpha}-1}{1-\alpha}, & 1 < \alpha, \quad x > 0 \end{cases} \tag{1}$$

The set of such functions is denoted by: $\mathcal{U}(DARA)$*,* $\mathcal{U}(CRRA)$*, respectively. We shall often abuse notation and write* $U(0)$ *for* $\lim_{W\to)0^+} U(W)$*, provided it exists.*

Remark 1.4. *Note that since the domain is bounded from below, the utility function may be bounded from below as well.*

Remark 1.5. *Note that* $\mathcal{U}(CRRA)$ *is a strict subset of* $\mathcal{U}(DARA)$*.*

Remark 1.6. *Note that the CRRA class is normalized such that all the functions in this class cross at the point* $(1, 0)$*. We then obtain log as the limiting case:* $\lim_{\alpha\to 1} \frac{x^{1-\alpha}-1}{1-\alpha} = \lim_{\alpha\to 1} \frac{-\log x}{-1} = \log x$*.*

Remark 1.7. *Note that we can shift the utility functions in* $\mathcal{U}(DARA)$ *to any interval* $(a, +\infty)$ *by defining* $U^* : (a, +\infty) \to \mathbb{R}$ *such that* $U^*(x) = U(x-a)$*. So the restriction on the domain of the function is really that it is bounded from below, not more that that.*

In what follows, we shall interpret S as the set of wealth levels and $\inf S = 0$ as wealth level associated with bankruptcy. We will denote the decision maker's initial total wealth as W and the net winnings from accepting a given risk as X. We require that all relevant wealth positions are in the set $S = (0, +\infty)$, i.e. we assume that the decision maker cannot accept negative wealth positions, below the bankruptcy level.

We now define buying and selling price for a lottery along the lines of Raiffa (1968) . We also define a riskiness measure which extends the one proposed in Foster and Hart (2009).

Unless otherwise stated, we assume that U satisfies Assumption 1.1 and is fixed. Given an agent with initial wealth level W, $S(X,W)$ is the minimal amount he is willing to accept in exchange for X. Similarly, $B(X,W)$ is the maximal amount he is willing to pay for X. Formally:

Definition 1.2. *Given an agent U with initial wealth level $W \in \mathbb{R}$ as well as lottery X, selling price of X, denoted by $S(X,W)$, and buying price of X, denoted by $B(X,W)$ are defined as real numbers satisfying:*

$$\mathbb{E}U[W+X] = U[W+S(X,W)] \tag{2}$$

$$\mathbb{E}U[W+X-B(X,W)] = U(W) \tag{3}$$

For a given lottery X we will be interested in selling and buying prices of X as functions of wealth. These will be denoted by $S(X,\cdot)$ and $B(X,\cdot)$, respectively.

Definition 1.3. *Given an agent U and a lottery $X \in \mathcal{X}_R$ the U-riskiness measure of X, denoted by $R(X)$ is defined as a real number satisfying:*

$$\mathbb{E}U\left[R(X)+X\right] = U[R(X)] \tag{4}$$

We wish to study the existence and uniqueness of measures defined above. Before we do it, we need some preliminary results. We first provide some basic equivalences between B, S and R provided they exist.

Proposition 1.1. *Suppose that $U \in \mathcal{U}(DARA)$ is fixed. Provided they exist and are unique, B, S and R satisfy the following equivalences.*

$$0 \equiv S[X-B(X,W),W] \tag{5}$$

$$B(X,W) \equiv S[X,W-B(X,W)] \tag{6}$$

$$S(X,W) \equiv B[X,W+S(X,W)] \tag{7}$$

$$W \equiv R[X-S(X,W)] - S(X,W) \tag{8}$$

$$W \equiv R[X-B(X,W)] \tag{9}$$

$$B[X,R(X-\Delta)] \equiv S[X,R(X-\Delta)-\Delta] \tag{10}$$

$$0 \equiv S[X,R(X)] \equiv B[X,R(X)] \tag{11}$$

Proof. The equivalences follow directly from the definitions. For example the first of the above equivalences is easily checked as follows:

$$\begin{aligned} U(W) &= \mathbb{E}U\{W + [X - B(X,W)]\} \\ &= \mathbb{E}U(W+Y) \\ &= U[W + S(Y,W)] \\ &= U\{W + S[X - B(X,W), W\} \end{aligned}$$

The condition (5) follows. The rest of proved similarly. □

2 Results

2.1 Existence and uniqueness

The following theorem due to Dybvig and Lippman (1983) implies that if $R(X)$ exists, then it is unique. Following Yaari (1969), for a lottery X and an agent U, define the acceptance set $A_X := \{W \in \mathbb{R} : \mathbb{E}U(W+X) > U(W)\}$ as the set of wealth levels at which the DM strictly prefers to accept the lottery.

Theorem 2.1 (Dybvig and Lippman (1983))**.** *Let U be a strictly increasing concave utility function with continuous second derivative. Then the absolute risk aversion ARA is nonincreasing if and only if for each gamble X, A_X is an interval of the form $(\theta_X, +\infty)$, where $-\infty \leq \theta_X \leq +\infty$.*

Proposition 2.1 (DARA)**.** *i. Bernoulli utility function exhibits DARA*

ii. For any lottery X and wealth W such that $B(X,W)$ and $S(X,W$ are defined, the following holds: $\frac{dB(X,W)}{dW} \in (0,1)$ and $\frac{dS(X,W)}{dW} > 0$. Moreover, for a non-degenerate lottery $X \in \mathcal{X}$, it holds:

$$B(X,W) > 0 \iff B(X,W) < S(X,W)$$

Proof. In Lewandowski (2013) and Lewandowski (2014).

Proposition 2.2. *Suppose that $U \in \mathcal{U}(DARA)$. Then for any nondegenerate $X \in \mathcal{X}$ there exist $A_l(X), A_u(X) \in (\min(X), \mathbb{E}(X))$ given by:*

$$\mathbb{E}U[-\min(X) + X] = U[-\min(X) + A_l(X)] \tag{12}$$

$$\mathbb{E}\exp(-\alpha X) = \exp[-\alpha A_u(X)] \tag{13}$$

where $\alpha = \lim_{W\to\infty} ARA(W)$, such that the functions:

$$S(X,\cdot) : (-\min(X), \infty) \to (A_l(X), A_u(X))$$
$$B(X,\cdot) : (-\min(X) + A_l(X), \infty) \to (A_l(X), A_u(X))$$

are strictly increasing and surjective. Moreover,

$$A_l(X) = \min(X) \iff \lim_{W\to 0^+} U(W) = -\infty \tag{14}$$

$$A_u(X) = \mathbb{E}(X) \iff \lim_{W\to +\infty} ARA(W) = 0 \tag{15}$$

Proof. We begin by proving the result for the selling price S and then use the equivalence (7) in Proposition 1.1 to extend it to the buying price B. If $U : (0, \infty) \to \mathbb{R}$ is continuous then for any lottery $X' := W + X$ there exists its certainty equivalent $\mathrm{CE}(X') := W + S$. If U is strictly increasing then the certainty equivalent of a lottery is unique. Since $\mathrm{dom}(U) = (0, \infty)$, a unique number $S(X, W)$ exists as defined by (2) for any $X \in \mathcal{X}$ iff $W \in (-\min(X), \infty)$. Since U is strictly concave then for any nondegenerate lottery X by Jensen's inequality $S(X, W), B(X, W) < \mathbb{E}(X))$. Since U is strictly increasing then by Expected Utility $S(X, W), B(X, W) > \min(X)$. Note that since $S(X, \cdot)$ is strictly increasing by Proposition 2.1, then $\arg\inf_W[S(X, W)] = A_l(X)$ is obtained for $W = -\min(X)$ and so it is given by (12). By the previous argument it must lie in $(-\min(X), \mathbb{E}(X))$. I now prove (14). Suppose that the RHS of the equivalence (14) does not hold so that $\lim_{W\to 0^+} U(W) = M > -\infty$. Then for any nondgenerate X, $A_l(X)$ cannot be equal to $\min(X)$ since in this case $\mathbb{E}U(-\min(X) + X) > U(-\min(X) + \min(X)) = M$ and the condition 12 does not hold. So $A_l(X)$ must be strictly greater than $\min(X)$. On the other hand, if $\lim_{W\to 0^+} U(W) = -\infty$, then $-\infty = \mathbb{E}U(-\min(X) + X) < U(-\min(X) + A_l(X))$ unless $A_l(X) = \min(X)$. Hence the equivalence (14) holds. We now show that $A_u(X)$ is given by (13). Suppose that $\lim_{W\to\infty} \mathrm{ARA}(W) = \alpha \geq 0$. Then we now that for any $\epsilon > 0$ there exist W_0 such that for all $W > W_0$, $\mathrm{ARA}(W) - \alpha < \epsilon$. So, as W tends to infinity the utility function U becomes closer and closer to the CARA utility with $ARA(W) = \alpha$. Such utility is, up to positive linear transformations, of the form $U(x) = 1 - \exp(-\alpha x)$. Hence, the limiting case of $S(X, W)$ as W tends to infinity is given by (13). I now prove (15). Suppose the RHS of the equivalence does not hold, so that $\lim_{W\to=\infty} \mathrm{ARA}(W) = \alpha > 0$. Using a Taylor series approximation around wealth W up to the second order, the LHS of equation (2) can be written as $\mathbb{E}[U(W) + U'(W)X + \frac{1}{2}U''(W)X^2] = U(W) + U'(W)\left[\mathbb{E}(X) - \frac{1}{2}\mathrm{ARA}(W)\mathbb{E}(X^2)\right]$. The RHS of (2), on the other hand, can be approximated by $U(W) + U'(W)S(X, W)$. So $S(X, W) \approx \mathbb{E}(X) - \frac{1}{2}\mathrm{ARA}(W)\mathbb{E}(X^2)$. Since $0 < \mathbb{E}(X^2) < \infty$ because X is nondegenerate and finitely supported, so we know that $\sup_W S(X, W) < \mathbb{E}(X)$ because $\mathrm{ARA}(W) = \alpha > 0$ for all W. So the $\Rightarrow$ direction of (15) is proved.

Now we prove the opposite direction. Suppose that $\lim_{W\to=\infty} \mathrm{ARA}(W) = 0$. Using (13) we write that $A_u(X) = -\frac{1}{\alpha} \ln\left[\mathbb{E}\exp(-\alpha X)\right]$. One can easily verify using the de l'Hospital rule that $\lim_{\alpha\to 0^+} A_u(X) = \mathbb{E}(X)$. So the equivalence (15) is proved. So we have proved the proposition for the selling price $S(X, W)$. By 7 one can easily verify the proposition for $B(X, W)$. □

Proposition 2.3. *Let $U \in \mathcal{U}(DARA)$. The following are equivalent:*

1. *The utility function satisfies:*

   a) U is unbounded from below.

   b) $\lim_{x\to+\infty} ARA(x) = 0$.

2. *for a nondgenerate $X \in \mathcal{X}$,*

$$S[X, (-\min(X), \infty)] = B[X, (0, \infty)] = (\min(X), \mathbb{E}(X)). \tag{16}$$

3. *a unique U-riskiness measure exists for all $X \in \mathcal{X}_R$.*

We will say that S and B that satisfy (16) have *full range.*

Proof. The equivalence between 1. and 2. is a corollary to proposition 2.2. That 1. implies 3. follows from the equivalence between 1. and 2. and (11). We have that 1. holds iff (16) holds. For $X \in \mathcal{X}_R$, $\inf_W S(X, W) = \min(X) < 0$ and $\sup_W S(X, W) = \mathbb{E}(X) > 0$. Since U is continuous, so is $S(X, \cdot)$. It is furthermore strictly increasing and hence there must exist a unique $W \in (-\min(X), \infty)$ such that $S(X, W) = 0$. By (11) it is the extended riskiness measure of X. We now prove that 3. implies 1. We now prove that 2. implies 1. Contrary to 1.a) Suppose that U is bounded from below. It means that $\lim_{W\to 0^+} U(W) = M$ for some $M > -\infty$. Consider $X = \langle -1, t; \epsilon, 1-\epsilon\rangle$. We require that $\epsilon > 0$ and $t > \frac{\epsilon}{1-\epsilon}$ so that $X \in \mathcal{X}_R$. Note that for any $\epsilon > 0$ the second requirement is satisfied by taking t large enough, which is possible as $\mathrm{dom}(U)$ does not have an upper bound. We know that since U is strictly increasing, $U(1+t) - U(1) = \delta > 0$. Then choose $epsilon > 0$ such that $\frac{\delta}{-M} < \frac{\epsilon}{1-\epsilon}$. Note that it is possible for any strictly increasing U. We have that:

$$\begin{aligned}
\lim_{W\to 1} \phi(X, W) &= \lim_{W\to 1} \epsilon U(W-1) + (1-\epsilon)U(1+t) - U(W) \\
&> \epsilon M + (1-\epsilon)\delta \\
&> 0
\end{aligned}$$

where the last inequality follows from our assumption. We now invoke (Dybvig and Lippman, 1983) theorem once again by which we know that for a nonincreasing ARA, $\phi(X, W) > 0$ for some W implies $\phi(X, W') > 0$ for any $W' > W$. Since we know $\phi(X, W) > 0$ for the lowest possible level of $W \to -\min(X)$, then it must be that $\phi(X, W) > 0$ for all $W \in (0, +\infty)$. Hence $R(X)$ does not exist.

Now contrary to 1.b) assume that $\lim_{x \to +\infty} \text{ARA}(x) = \delta > 0$. Since U is DARA, it follows that $\text{ARA}(x) \geq \delta$ for all x. Then choose $\epsilon \in (0, 1)$ such that $\delta > \frac{2\epsilon}{1+\epsilon^2}$. It always can be done because for the RHS of this inequality tends to 0 as ϵ does. Now consider the lottery $X = \langle -1+\epsilon, 1+\epsilon; 0.5, 0.5 \rangle$. Note that $\mathbb{E}(X) = \epsilon > 0$ and $\mathbb{P}(X < 0) = 0.5 > 0$ since *epsilon* < 1, so that $X \in \mathcal{X}_R$ Now we approximate using second order Taylor expansion around W:

$$
\begin{aligned}
\phi(X, W) &= \mathbb{E}\left[U(W) + XU'(W) + \tfrac{1}{2}U''(W)X^2\right] - U(W) \\
&= U'(W)\left[\epsilon - \tfrac{1}{2}\text{ARA}(W)(1+\epsilon^2)\right] \\
&< 0.
\end{aligned}
$$

where the last inequality follows from $\text{ARA}(W) > \frac{2\epsilon}{1+\epsilon^2}$. So X is rejected at any wealth level and $R(X)$ does not exist. □

Remark 2.1. *If U belongs to $\mathcal{U}(DARA)$ and satisfies the conditions stated in 1.a) and 1.b) in proposition 2.3 then for any $X \in \mathcal{X}$, $X - s \in \mathcal{X}_R$ if and only if $s = S(Y, W)$ for some $W \in domS(X, \cdot)$. On the other hand, if the conditions stated in 1.a) or 1.b) are violated, then "if and only if" statement in the above equivalence should be replaced with the "if" statement only. Obviously, S in the above statement can be replaced with B.*

Remark 2.2. *It is worth noting that the condition $\lim_{W \to 0^+} ARA(W) = +\infty$ is not sufficient to ensure the existence of $R(X)$ for any $X \in \mathcal{X}_R$. One needs stronger requirement that the utility is unbounded from below. As a counter example, consider $U(W) = W^{0.5}$. It is clear that for this utility function $\lim_{W \to 0^+} ARA(W) = +\infty$. However, consider $X = (+500, -100, 0.5, 0.5)$. It clearly belongs to $\mathcal{X}_R$. However, it is easily verifiable that for $W \in (100, \infty)$, $\mathbb{E}U(W + X) - U(W) > 0$, and hence $R(X)$ does not exist.*

Definition 2.1. *For $X \in \mathcal{X}$, $s, b \in rng[S(X, \cdot)] = rng[B(X, \cdot)]$, define $\phi(X, s), \psi(X, b) \in \mathbb{R}$ such that $\phi(X, s) = R(X - s) - s$ and $\psi(X, b) = R(X - b)$.*

The above definitions will be useful in establishing the relationship between S and B on the one hand, and R on the other. $\phi(X, \cdot)$ provides the inverse of the $S(X,\cdot)$ and $\psi(X,\cdot)$ provides the inverse of $B(X,\cdot)$. We shall analyze mostly the selling price S and ϕ and infer the corresponding properties of B using the equivalence (7) and of ψ using the fact that $\psi(X,b) = \phi(X,b) + b$. We first begin by stating some equivalences that will be used later on. For any lottery X and $W \in \text{dom}S(X,\cdot)$, $s \in \text{rng}S(X,\cdot)$ the following holds:

$$W \equiv \phi[X, S(X,W)] \tag{17}$$

$$s \equiv S[X, \phi(X,s)] \tag{18}$$

2.2 Main result

Proposition 2.4. *Let U be a function satisfying the assumptions stated in the introduction. For $X, Y \in \mathcal{X}$, let $A = domS(X,\cdot) \cap domS(Y,\cdot)$ and $B = rngS(X,\cdot) \cap rngS(Y,\cdot)$. The following relationships hold:*

1. *$R(X-s) = R(Y-s)$ for some $s \in B$ if and only if $S(X,W) = S(Y,W) = s$ for some $W \in A$.*

2. *If $R(Y-s) > R(X-s)$ for some $s \in B$, then $S(X,W) > S(Y,W)$ for $W \in [R(X-s)-s, R(Y-s)-s] \cap A$.*

3. *If $S(X,W) > S(Y,W)$ for some $W \in A$, then $R(Y-s) > R(X-s)$ for $s \in [S(Y,W), S(X,W)] \cap B$.*

Proof. We first prove 1. Let $X, Y \in \mathcal{X}$. Assume that $R(X-s) = R(Y-s)$ for $s \in B$. This is equivalent to $\phi(X,s) = \phi(Y,s) = W$ for $W \in A$. And this in turn is equivalent to $S(X,W) = S(Y,W) = s$ for $s \in B$.

We now prove 2. Assume that $R(Y-s) > R(X-s)$ for some $s \in B$. Using the definition of ϕ this is equivalent to $\phi(Y,s) > \phi(X,s)$. We know that $\phi(Y,s) \in A$ or $\phi(X,s) \in A$ because $s \in B$. Since $A = [a,\infty)$ for $a > -\infty$, we know that $\phi(Y,s) \in A$. Take $W = \phi(Y,s)$. By definition of ϕ, this is equivalent to $s = S(Y,W)$. Using the equivalence (17) and rewriting we get:

$$\begin{aligned} &\phi(Y,s) > \phi(X,s), \text{ for } s \in B \\ \iff \phi[X, S(X,W)] = W = \phi[Y, S(Y,W)] > \phi[X, S(Y,W)] \end{aligned}$$

Since for any $X \in \mathcal{X}$, $S(X,\cdot)$ is strictly increasing, so is its inverse $\phi(X,\cdot)$. Using this fact, the above inequality is equivalent to $S(X,W) > S(Y,W)$. Now take any $V < W$ such that $V \in A$ and $V \geq \phi(X,s)$. Since $S(Y,\cdot)$ and $S(X,\cdot)$ are strictly increasing, for any such V, $S(X,V) \geq S(X,\phi(X,s)) = s = S(Y,W) > S(Y,V)$. This concludes the proof of 2.

We now prove 3. Suppose $S(X,W) > S(Y,W)$ for $W \in A$. Since $W \in A$, then it must be that $S(X,W) \in B$ or $S(Y,W) \in B$. We consider three cases: Suppose only $S(X,W) \in B$. Let $s = S(X,W)$, which by definition of ϕ, is equivalent to $\phi(X,s) = W$. Then using the equivalence (18) we have:

$$\begin{aligned} & S(X,W) > S(Y,W), \text{ for } W \in A \\ \iff & S[Y,\phi(Y,s)] = s = S[X,\phi(X,s)] > S[Y,\phi(X,s)] \end{aligned}$$

So by the strict monotonicity of $S(Y,\cdot)$, we have that $\phi(Y,s) > \phi(X,s)$ or, by the definition of ϕ, $R(Y-s) > R(X-s)$. Now take $t < s$ such that $t \in B$ and $t \geq S(Y,W)$. By strict monotonicity of $\phi(X,\cdot)$ and $\phi(Y,\cdot)$, for all such t, $\phi(Y,t) \geq \phi[Y,S(Y,W)] = W = \phi[X,S(X,W)] = \phi(X,s) > \phi(X,t)$, and so $\phi(Y,t) > \phi(X,t)$. This concludes the proof for the first case. The second case is when only $S(Y,W) \in B$. Let $s = S(Y,W)$, so that $\phi(Y,s) = W$. Then, by similar arguments as before:

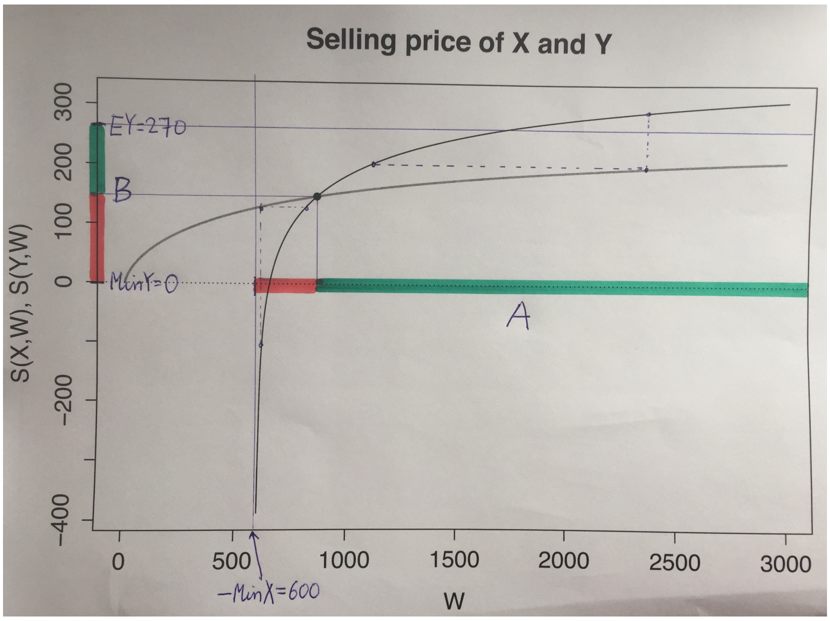
$$\begin{aligned} & S(X,W) > S(Y,W), \text{ for } W \in A \\ \iff & S[X,\phi(Y,s)] > S[Y,\phi(Y,s)] = s = S[X,\phi(X,s)] \end{aligned}$$

So $\phi(Y,s) > \phi(X,s)$ or $R(Y-s) > R(X-s)$. Take $t > s$ such that $t \in B$ and $t \leq S(X,W)$. For all such t, by strict monotonicity of $\phi(X,\cdot)$ and $\phi(Y,\cdot)$, it holds: $\phi(X,t) \leq \phi[X,S(X,W)] = W = \phi[Y,S(Y,W)] = \phi(Y,s) < \phi(Y,t)$, and so $\phi(X,t) < \phi(Y,t)$. This concludes the second case. The final case is when both $S(X,W) \in B$ and $S(Y,W) \in B$. In this case we can choose s to be either one of the two. We assume that $s = S(X,W)$ and so $\phi(X,s) = W$. By similar arguments as above we obtain that $\phi(X,s) < \phi(Y,s)$. Now take $t < s$ such that $t \geq S(Y,W)$. We know that $t \in B$ since B is connected and both $S(X,W)$ and $S(Y,W)$ belong to to B. We then have by strict monotonicity of $\phi(X,\cdot)$ and $\phi(Y,\cdot)$, that $\phi(Y,t) \geq \phi[Y,S(Y,W)] = W = \phi[X,S(X,W)] = \phi(X,s) > \phi(X,t)$, and so $\phi(Y,t) > \phi(X,t)$. This concludes the third case and the proof of 3. □

The following are further properties of S, B and R:

Proposition 2.5. *For any $U \in \mathcal{U}(DARA)$, any $X \in \mathcal{X}$, and any $\Delta, W \in \mathbb{R}$, the following holds:*

$$\begin{aligned} R(X+\Delta) &\leq R(X) - \Delta \iff \Delta \geq 0 \\ B(W, X+\Delta) &= B(X,W) + \Delta \qquad (19) \\ S(W, X+\Delta) &= S(W+\Delta, X) + \Delta \qquad (20) \end{aligned}$$



Remark 2.3. *Note that R does not satisfy the translation invariance property. However, if for a given U and W we define $\theta(X) := -B(X, W)$ as a measure of risk, then this measure satisfies translation invariance: $\theta(X + \Delta) = \theta(X) - \Delta$.*

3 Riskiness measure of Foster-Hart

Foster and Hart (2009) define an operational measure of riskiness as follows. The initial wealth is $W_1 > 0$. At every period $t = 1, 2, ...$, the decision maker with wealth W_t is offered a gamble $\mathbf{x_t}$. He may accept or reject the gamble. His wealth next period is $W_{t+1} = W_t + \mathbf{x_t}$ if he accepts and $W_{t+1} = W_t$ if he rejects. Simple strategy of the decision maker whether to accept gamble $\mathbf{x_t}$ at time t or not is assumed to be stationary Markov strategy - it depends only on the gamble $\mathbf{x_t}$ and current wealth level W_t. Simple strategy is homogeneous or scale-invariant if "accept X at W" implies "accept λX at λW", for any $\lambda > 0$.

If borrowing is not allowed, bankruptcy occurs when wealth converges to zero as time goes to infinity. A given strategy s yields no-bankruptcy for the process $(\mathbf{x_t})_{t=1,2,...}$ and the initial wealth W_1 if probability of bankruptcy is zero, i.e. $\mathbb{P}[\lim_{t\to\infty} W_t = 0] = 0$. Strategy guarantees no-bankruptcy if it yields no-bankruptcy for every process $(\mathbf{x_t})_{t=1,2,...}$ and every initial wealth level W_1. The technical assumptions state that gambles are assumed to be finite-valued, with finite support and such that $\mathbb{E}[X] > 0$ and $\mathbb{P}[X < 0] > 0$, where $\mathbb{P}[E]$ denotes a probability of an event E (positive expected value and losses are possible). The stochastic process $(\mathbf{x_t})_{t=1,2,...}$ is assumed to be finitely generated.

The main theorem of Foster and Hart (2009) states the following.

Theorem 3.1 (Foster and Hart (2009))**.** *For every gamble X there exists a unique real number $R_{FH}(X) > 0$ such that: a homogeneous strategy s guarantees no-bankruptcy if and only if for every gamble X and wealth $W > 0$,*

$$W < R_{FH}(X) \;\Rightarrow\; s \text{ rejects } X \text{ at } W$$

Moreover, $R_{FH}(X)$ satisfies the following equation

$$\mathbb{E}\left[\log\left(1 + \frac{X}{R_{FH}(X)}\right)\right] = 0 \tag{21}$$

Foster and Hart (2009) call $R_{FH}(X)$ the measure of riskiness of X.

There is a link between the riskiness measure and expected utility maximizing individuals. Consider an expected-utility maximizer with utility function U:

$$\text{accept } X \text{ at } W \iff \mathbb{E}U(W + X) \geq U(W) \tag{22}$$

Notice that for the logarithmic utility function I can rewrite condition on the RHS of (22) in relative - instead of absolute - terms, as follows:

$$\mathbb{E}\left[\log\left(1 + \frac{X}{W}\right)\right] \geq 0$$

It is clear that the index $R_{FH}(X)$ has the property that the logarithmic utility rejects X if $W < R_{FH}(X)$ and accepts X if $W \geq R_{FH}(X)$. Hence by the theorem above logarithmic utility represents a strategy that is among those which guarantee no-bankruptcy.

3.1 Extension to DARA

Before I will proceed to the next subsection, I want to demonstrate that for a certain class of DARA utility functions which are not necessarily CRRA, no-bankruptcy is guaranteed. First I will need the following lemma, which is also of interest for its own sake.

Without loss of generality[1] assume that utility function U satisfies the following: $U(1) = 0$ and $U'(1) = 1$. Given such utility function U define relative risk aversion function as $RRA(x) = -\frac{U''(x)x}{U'(x)}$. For utility function which is denoted U_i I will use notation RRA_i for the corresponding relative risk aversion function. Then the following lemma is true.

[1] Cardinal utility function is unique only up to affine transformation.

Lemma 3.1. *For some $\delta > 0$, suppose that $RRA_i(y) > RRA_j(y)$ for all y such that $|y| < \delta$. Then $U_i(y) < U_j(y)$ whenever $y \neq 1$ and $|y| < \delta$*

Proof. First, let me say that the proof is very similar to that used in lemma 2 of Aumann and Serrano (2008). They prove a similar proposition for absolute risk aversion.
Let $|y| < \delta$. If $y > 1$, then

$$\begin{aligned}
\log U_i'(y) &= \log U_i'(y) - \log U_i'(1) \\
&= \int_1^y [\log U_i'(z)]' dz \\
&= \int_1^y \frac{U_i''(z)}{U_i'(z)} dz \\
&= -\int_1^y \frac{RRA_i(z)}{z} dz \\
&< -\int_1^y \frac{RRA_j(z)}{z} dz = \log U_j'(y)
\end{aligned}$$

If $0 < y < 1$, then

$$\begin{aligned}
\log U_i'(y) &= \log U_i'(y) - \log U_i'(1) \\
&= -\int_y^1 [\log U_i'(z)]' dz \\
&= -\int_y^1 \frac{U_i''(z)}{U_i'(z)} dz \\
&= \int_y^1 \frac{RRA_i(z)}{z} dz \\
&> \int_y^1 \frac{RRA_j(z)}{z} dz = \log U_j'(y)
\end{aligned}$$

Hence $\log U_i'(y) \lesseqgtr \log U_j'(y)$, when $y \gtreqless 1$. It follows that $U_i'(y) \lesseqgtr U_j'(y)$, when $y \gtreqless 1$.
If $y > 1$, then

$$U_i(y) = \int_1^y U_i'(z) dz < \int_1^y U_j'(z) dz = U_j(y)$$

If $0 < y < 1$, then

$$U_i(y) = -\int_y^1 U_i'(z) dz < -\int_y^1 U_j'(z) dz = U_j(y)$$

And hence the lemma is proved. □

Equipped with lemma 3.1 I can now demonstrate for which DARA utility functions in general the condition of no-bankruptcy is guaranteed.

Proposition 3.1. *For all bounded-valued lotteries and for all DARA utility functions for which $RRA(x) \geq 1$, $\forall x \in D$, where $RRA(x)$ is relative risk aversion function evaluated at x and D is the utility function's domain, no-bankruptcy is guaranteed.*

Proof. No-bankruptcy is guaranteed for logarithmic utility function for which relative risk aversion coefficient is equal to one. Take a DARA utility function U for which relative risk aversion is not less than one for all arguments in the domain of U. For any wealth level W I can normalize U without loss of generality so that $U(W) = \log(W)$. By lemma 3.1, since $RRA(y) \geq 1$ for all finite y, it is true that $U(y) \leq \log(y)$ and by normalization $U(W) = \log(W)$. It follows that if logarithmic utility function "rejects" a lottery X, utility U also "rejects" this lottery. And hence it also guarantees no-bankruptcy. □

4 Buying/selling price and the riskiness measure for the special case of CRRA

The proposition below establishes the range, domain and some properties of the selling and buying price for a lottery as functions of wealth for the CRRA class of utility functions.

Proposition 4.1 (CRRA)**.** *Given the class of CRRA utility functions defined by (1), for any non-degenerate lottery $X \in \mathcal{X}$ the functions $S(\cdot, X)$ and $B(\cdot, X)$:*

a) are strictly increasing and strictly concave,

c) are one-to-one functions with the following domains and ranges:

- *The case of $\alpha > 1$ (including the limiting case $\alpha \to 1$):*

$$\text{dom}[S(\cdot, X)] = (-x_1, +\infty) \quad \text{ran}[S(\cdot, X] = (x_1, \mathbb{E}[X])$$
$$\text{dom}[B(\cdot, X)] = (0, +\infty) \quad \text{ran}[B(\cdot, X)] = (x_1, \mathbb{E}[X])$$

- *The case of $\alpha \in (0, 1)$*

$$\text{dom}[S(\cdot, X)] = (-x_1, +\infty) \quad \text{ran}[S(\cdot, X)] = (W_\alpha(X) + x_1, \mathbb{E}[X])$$
$$\text{dom}[B(\cdot, X)] = (W_\alpha(X), +\infty) \quad \text{ran}[B(\cdot, X)] = (W_\alpha(X) + x_1, \mathbb{E}[X])$$

where $W_\alpha(X) = U_\alpha^{-1}[\mathbb{E}U_\alpha(-x_1 + X)]$ which is finite for the case of $\alpha \in (0, 1)$.

Proof. Strict monotonicity follows from proposition 2.1 above. The domain and range of the above functions is established in Lewandowski (2014). This result taken together with strict monotonicity implies that the functions are one-to-one. Strict concavity is proved in Lewandowski (2013).

I define measure R for lottery X for CRRA utility function. This measure should satisfy the following condition:

$$\frac{1}{1-\alpha}\mathbb{E}\left(1+\frac{X}{R(X)}\right)^{1-\alpha}-\frac{1}{1-\alpha}=0 \tag{23}$$

for a given lottery X and coefficient α. I want to ensure that such measure is well defined and unique. As already proved in the previous subsection measure of riskiness is unique if it exists. The necessary conditions are already provided in the former subsection and in particular, I will focus only on non-degenerate n-dimensional lotteries X with bounded values[2] such that $\mathbb{P}[X<0]>0$ and $\mathbb{E}(X)>0$. Furthermore, I will restrict attention only to wealth levels W, such that $W \geq L(X) > 0$. The fact that $L(X)>0$ follows from the fact that X may take negative values. Define lottery $Y = 1+\frac{X}{W}$. Notice that this lottery takes only non-negative values. It takes the lowest value of zero for $x_i = -L(X)$ for some $i \in \{1,...,n\}$, since $W \geq L(X)$.

Notice that for the function form above, the following is true: $U(1)=0$, $U'(y)=y^{-\alpha}$ and $U'(1)=1$. Suppose there are two different CRRA utility functions with relative risk aversion coefficients equal to α_i and α_j, respectively. Suppose further that $\alpha_i > \alpha_j$. Then from lemma 3.1 I know that $U(y,\alpha_i) < U(y,\alpha_j)$, for $y \in [0,\delta)$, some $\delta > 0$ and $y \neq 1$. Hence,

$$\begin{aligned} & \frac{1}{1-\alpha_i}\mathbb{E}\left(1+\frac{X}{R(X)}\right)^{1-\alpha_i}-\frac{1}{1-\alpha_i} \\ < \quad & \frac{1}{1-\alpha_j}\mathbb{E}\left(1+\frac{X}{R(X)}\right)^{1-\alpha_j}-\frac{1}{1-\alpha_j} \end{aligned}$$

Let's define the following function:

$$\phi(\lambda,\alpha) = \frac{1}{1-\alpha}\sum_{i=1}^{n} p_i[1+\lambda x_i]^{1-\alpha} - \frac{1}{1-\alpha} \tag{24}$$

$$0 \leq \lambda \leq \frac{1}{L(X)}, \quad x_i \in [-L(X), +M(X)]$$

where $M(X)$ is the maximal gain in X and $L(X)$ is the maximal loss of X, both assumed to be finite.

I want to find out whether this function has a unique $\lambda > 0$, for which this function is equal to zero, given α, and whether it has a unique α for which the function is equal to zero, given that $\lambda = \frac{1}{L(X)}$. It turns out that the answer to both questions is positive, as I will demonstrate below.

[2] The following condition holds: there exists $\delta > 0$ such that $|x_i| < \delta \;\; \forall i \in \{1,...,n\}$.

Lemma 4.1. *The following properties characterize function ϕ:*

$$
\begin{aligned}
&\phi(0,\alpha) = 0\\
&\frac{\partial\phi(\lambda,\alpha)}{\partial\lambda} = \sum_{i=1}^{n} p_i[x_i(1+\lambda x_i)^{-\alpha}]\\
&\left.\frac{\partial\phi(\lambda,\alpha)}{\partial\lambda}\right|_{\lambda=0} = \sum_{i=1}^{n} x_i p_i = \mathbb{E}[X] > 0\\
&\frac{\partial^2\phi(\lambda,\alpha)}{\partial^2\lambda} = \alpha\sum_{i=1}^{n} p_i x_i^2(1+\lambda x_i)^{-\alpha-1} < 0 \quad \text{for} \quad \alpha > 0\\
&\lim_{\lambda\to\frac{1}{L(X)}}\lim_{\alpha\to 1}\phi(\lambda,\alpha) = -\infty\\
&\lim_{\lambda\to\frac{1}{L(X)}}\phi(\lambda,0) = \lim_{\lambda\to\frac{1}{L(X)}}\sum_{i=1}^{n} p_i(1+\lambda x_i) - 1 = \frac{1}{L(X)}\mathbb{E}[X] > 0 \qquad (25)
\end{aligned}
$$

Furthermore $\lim_{\lambda\to\frac{1}{L(X)}}\phi(\lambda,\alpha)$ is a continuous function of α and it is strictly monotonic in α (see lemma 3.1). Therefore the following result holds:

Proposition 4.2. *Given function $\phi(\lambda,\alpha)$ and a random variable X with n values denoted by x_i for $i = 1, ..., n$, where $\mathbb{E}(X) > 0$ and $\mathbb{P}[X < 0] > 0$, the following is true. Denote $L = L(X)$ and $M = M(X)$.*

$$
\exists!\alpha^* < 1 : \begin{cases} \alpha < \alpha^* & \phi(\frac{1}{L},\alpha) > 0 \\ \alpha = \alpha^* & \phi(\frac{1}{L},\alpha) = 0 \\ \alpha > \alpha^* & \phi(\frac{1}{L},\alpha) < 0 \end{cases}
$$

Furthermore, suppose I take $\alpha > \alpha^*$ and fix it. Then:

$$
\exists!\lambda^* : \begin{cases} \lambda < \lambda^* & \phi(\lambda,\alpha) > 0 \\ \lambda = \lambda^* & \phi(\lambda,\alpha) = 0 \\ \lambda > \lambda^* & \phi(\lambda,\alpha) < 0 \end{cases}
$$

Proof. Follows from the above stated properties of a function ϕ (lemma 4.1). □

The above proposition states that riskiness measure for CRRA is defined for $\alpha \geq \alpha^*$, where α^* depends on a lottery. In this case the riskiness measure is unique. For different α's from the set of α's satisfying $\alpha > \alpha^*$ I get different λ^*, which is the inverse of the riskiness measure. Let's define a function $\lambda^*(\alpha)$, where $\alpha > \alpha^*$ and $\phi(\lambda^*(\alpha),\alpha) = 0$. I have the following proposition:

Proposition 4.3. *The function $\lambda^*(\alpha)$ is decreasing in α.*

Proof. Suppose $\alpha_1 > \alpha_2$ and that $\alpha_1 > \alpha^*$. Then

$$\begin{aligned} 0 &= \phi(\lambda^*(\alpha_1), \alpha_1) \\ &= \frac{1}{1-\alpha_1} \sum_{i=1}^{n} p_i (1 + \lambda^*(\alpha_1) x_i)^{1-\alpha_1} - \frac{1}{1-\alpha_1} \\ &< \frac{1}{1-\alpha_2} \sum_{i=1}^{n} p_i (1 + \lambda^*(\alpha_1) x_i)^{1-\alpha_2} - \frac{1}{1-\alpha_2} = \phi(\lambda^*(\alpha_1), \alpha_2) \end{aligned}$$

Hence:

$$\phi(\lambda^*(\alpha_1), \alpha_2) > 0$$
$$\phi(\lambda^*(\alpha_2), \alpha_2) = 0$$

Since $\phi(\lambda, \alpha)$ is concave in λ and $\phi(\frac{1}{L}, \alpha) < 0$, I conclude that $\lambda^*(\alpha_2) > \lambda^*(\alpha_1)$. □

The above proposition states that the higher is α, the relative risk aversion coefficient, the higher is riskiness measure, which is the inverse of $\lambda^*(\alpha)$. It confirms a conjecture that since rejecting for wealth being below riskiness measure based on $\alpha = 1$ (Foster and Hart (2009) riskiness measure) guarantees no bankruptcy, also rejecting for wealth below riskiness measure based on $\alpha > 1$ guarantees no bankruptcy, as it means more rejection. To illustrate the above propositions and clarify the meaning of the different concepts and variables, look at the graph below:
This graph depicts the shape of $\phi(\lambda, \alpha)$ function for different values of relative risk aversion α within the CRRA class of utility functions. For α between 0 and α^* an extended riskiness measure is not defined since in this case function ϕ does not cross the zero axis. An extended riskiness measure is defined if $\alpha \geq \alpha^*$. Furthermore, it is also clear from the picture that an extended riskiness measure for values of α greater than 1 is necessarily greater than $R_{FH}(X)$ and hence rejecting X at wealth smaller than the extended riskiness measure in this case also guarantees no-bankruptcy.

5 Concluding remarks

In this paper I analyzed riskiness measure as introduced by Foster and Hart (2009). I gave simple intuition behind their result and I tried to make some steps towards extending this measure in two respects - first to define an extended riskiness measure based on DARA utility functions and derive necessary and sufficient conditions for existence and uniqueness of such measure for DARA

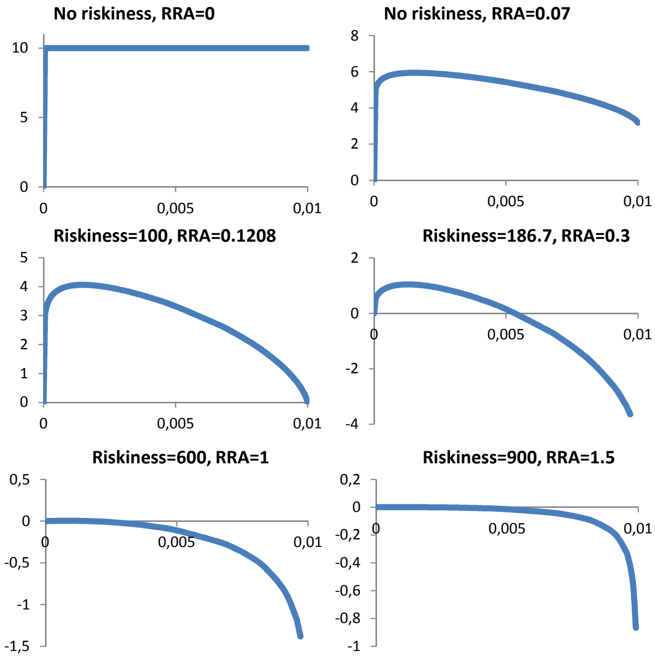


Figure 1: An extended riskiness measure for CRRA utility

and CRRA class of utility functions. Obviously, for the more specialized case of CRRA utility functions more exact conditions are obtained than for the more general case of DARA utilities. I also tried to extend the domain of riskiness measure. For gambles with non-positive expectation or no losses I proposed a way to compare their riskiness by subtracting prices from them. If the riskiness ordering is unchanged over the whole range of prices for which the lottery minus the price exists is unchanged, something can be inferred about the riskiness of a gamble without prices. To this end a number of useful properties relating buying and selling price for a lottery and riskiness measure were established and should be useful also for their own sake. An extension of Pratt (1964) famous result on comparative risk aversion involving riskiness measure along with buying and selling price for a lottery was stated and proved. Finally a simple link between decision-making using riskiness measure and decision-making using buying and selling price was developed.

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Appendix

Lemma 5.1. *Given any lottery X and wealth level W, the following three relations between buying price and selling price hold:*

$$\begin{aligned} S[W, X - B(X,W)] &= 0 && (26) \\ S[W - B(X,W), X] &= B(X,W) && (27) \\ B[W + S(X,W), X] &= S(X,W) && (28) \end{aligned}$$

Proposition 5.1. *For any non-degenerate lottery X and any wealth W such that buying and selling price exist, $S(X,W)$ and $B(X,W)$ lie in the interval $(\min(X), \mathbb{E}(X))$. For a degenerate lottery X, $S(X,W) = B(X,W) = x$.*

The following is a corollary to Pratt (1964) famous theorem of comparative risk aversion.

Corollary 5.1. *For a strictly increasing and twice differentiable utility function U with continuous second derivative, the following holds:*

- *$S(X,W)$ is increasing/constant/decreasing in W for every X iff $A(W)$ is decreasing/constant/increasing in W*

Proposition 5.2. *For any lottery X and any wealth W, for utilities with decreasing absolute risk aversion (DARA) the following equivalence holds:*

$$B(X,W) > 0 \iff B(X,W) < S(X,W)$$

Notice that for DARA utility function and $B(X,W) > 0$ the above result together with proposition 5.2 implies the following:

$$S(W, X+\Delta) - B(W, X+\Delta) = S(W+\Delta, X) - B(X,W) > S(X,W) - B(X,W)$$

Proposition 5.3. *For a strictly increasing and twice differentiable utility function U with continuous second derivative, the following holds:*

- *$B(X,W)$ is increasing/constant/decreasing in W for every X iff $A(W)$ is decreasing/constant/increasing in W*

Lemma 5.2. *For differentiable DARA utility functions, given any n-dimensional non-degenerate lottery X and any wealth level W, the following holds:*

- $\mathbb{E}U'(W+X) - U'(W + S(X,W)) > 0$
- $\mathbb{E}U'(W + X - B(X,W)) - U'(W) > 0$

- $\mathbb{E}U'(R(X)+X) - U'(R(X)) > 0$
- $0 < \frac{\partial B(X,W)}{\partial W} < 1$

Proof. From the definition of buying, selling price and the fact that they are both increasing in wealth, it follows that:

$$\frac{\partial S(X,W)}{\partial W} = \frac{\mathbb{E}U'(W+X) - U'(W+S(X,W))}{U'(W+S(X,W))} > 0$$
$$\frac{\partial B(X,W)}{\partial W} = \frac{\mathbb{E}U'(W+X-B(X,W)) - U'(W)}{\mathbb{E}U'(W+X-B(X,W))} > 0$$

All of the properties above follow immediately. □

Proposition 5.4. *For two different utility functions U_1 and U_2, any wealth level W and any n-dimensional non-degenerate random variable X with bounded values, I define corresponding selling and buying prices $S_1(W,X)$, $B_1(W,X)$ and $S_2(W,X)$, $B_2(W,X)$. The following equivalence holds:*

$$\forall W\ \forall X:\ \exists\ \delta > 0\ |x_i| < \delta\ \forall i \in \{1,...,n\}$$
$$S_1(W,X) > S_2(W,X) \iff B_1(W,X) > B_2(W,X)$$

Proposition 5.5. *The following two statements are equivalent:*

i. Bernoulli utility function exhibits CRRA

ii. buying and selling price for any lottery are homogeneous of degree one i.e.

$$S(\lambda W, \lambda X) = \lambda S(X,W),\ \forall \lambda > 0$$
$$B(\lambda W, \lambda X) = \lambda B(X,W),\ \forall \lambda > 0$$