# Some implications of Reference-Dependent Subjective Expected Utility model

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**Abstract**

Rabin and Thaler (2001) declared Expected Utility an ex-hypothesis or a dead parrot alluding to the famous sketch from Monthy Pythons Flying Circus. Following Cox and Sadiraj (2006) and others, one should distinguish between Expected Utility (EU) theory (a purely mathematical theory based on axioms) and Expected Utility models (EU theory plus a given economic interpretation). The most prevalent EU model is one that assumes consequentialism (Rubinstein, 2012). Consequentialism states that the decision maker has a single binary preference relation comparing probability distributions over final wealth levels. Preference relations over wealth changes for different levels of wealth are derived from this single preference relation. EU theory plus consequentialism is referred to as the standard EU model. It is argued that most of the critique against EU is against the standard EU model, or against consequentialism. We replace consequentialism with reference-dependence, retaining the EU hypothesis. Using Sugden (2003) framework, we show that many violations of the standard EU model can be explained assuming this different intrerpretation. Among the topics considered are: WTA/WTP disparity, preference reversal, complementary symmetry, preference homogeneity, loss aversion, reflection effect and the coexistence of insurance and gambling.

**Keywords:** preference reversal, WTA-WTP gap, complementary symmetry, loss aversion, reference dependence

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## 1 Introduction

Since 1738 Expected Utility (EU) hypothesis is an accepted paradigm of decision making under risk, applied successfully in a wide spectrum of economic analysis. Yet starting with Allais (1953) a bunch of contrary evidence against the standard EU model accummulated, culminating with (Rabin and Thaler, 2001, p.230) who, alluding the famous sketch from Monthy Pythons Flying Circus, stated: *We feel much like the customer in the pet shop, beating away a **dead parrot**.* And also:*(...) it is time for economists to recognize that expected utility is an **ex-hypothesis**, so that we can concentrate our energies on the important task of developing better descriptive models of choice under uncertainty.* The proponents of this view suggest replacing EU with Prospect Theory (Kahneman and Tversky, 1979, Tversky and Kahneman, 1992, Schmidt et al., 2008). Following another classic, I shall argue in the paper that *the reports of parrots death were an exaggeration.* I now sketch the argument.

Following Cox and Sadiraj (2006) it is important to distinguish between EU Theory (mathematical theory based on axioms) and an EU model (EU Theory plus a specific interpretation). As argued by Rubinstein (2012), the prevailing interpretation is that of consequentialism; it states that the decision maker with initial wealth level $W$ derives her risk preferences over wealth changes $\succsim_W$ from a single risk preference over final wealth levels $\succsim$ by taking: $l_1 \succsim_W l_2 \iff W + l_1 \succsim W + l_2$, where $l_1, l_2$ are two probability distributions over income – extra money beyond my wealth. EU Theory combined with consequentialism (or what Cox and Sadiraj (2006) call EU of terminal wealth model) is thus the standard EU model.

The first important observation is that most of the available experimental evidence, referred to as EU paradoxes, is evidence against the standard EU model. For example consider Rabin paradox. It states that in the EU model:

1. if the DM rejects an equal chance of getting \$110 or losing \$100 at all initial wealth levels,
2. then she must also reject an equal chance of getting an arbitrarily high sum of money or losing \$1000 at all initial wealth levels.

This is called a paradox because the premise that seems reasonable leads to a conclusion that does not. Rabin (2000) attributes this implausible implication to EU theory. However, it has been argued in a series of papers that this should rather be attributed to consequentialism and not to EU theory (Rubinstein, 2006, Cox and Sadiraj, 2006, Palacios-Huerta and Serrano, 2006, Foster and Hart, 2009 and Lewandowski, 2014).

This leads to a second important observation, namely that at least some of the evidence presented as contrary to EU is in fact evidence against the consequentialist interpretation.

There are alternatives to consequentialism. Expected Utility may be combined with *mental accounting*, such that the decision maker integrates his wealth only within a certain budget (Lewandowski, 2014); for example the decision maker may be EU maximizer within the restricted domain of his gambling activity, for the purposes of which he designates part of his wealth, called a gambling wealth. In the Rabin example above, if my gambling wealth is small (say \$150), it may be reasonable to reject the first gamble, because of the fear to run out of (gambling) money. Another alternative to consequentialism is *range-dependence* (Kontek and Lewandowski, 2016). The decision maker is an EU maximizer only when evaluating lotteries having the same range of lottery consequences (the interval between the lowest and the highest consequence in the support of the lottery).

The most well-known alternative to consequentialism, however, is *reference-dependence*, and this will also be the main focus of this paper. It generalizes consequentialism by allowing the decision maker's preferences over prospects defined on wealth changes to be inconsistent with her preferences over prospects defined over final wealth levels.

The general idea is to retain the EU hypothesis but to change the interpretation of the standard EU model by replacing consequentialism (final wealth interpretation) with reference-dependence (changes of wealth), where the reference point is assumed to be *status quo* and is allowed to be random. I demonstrate that the resulting model, the Expected Utility of wealth changes model, accommodates most of the known evidence against the standard EU model. In particular I analyze the following: WTA/WTP disparity, preference reversal, complementary symmetry, preference homogeneity, loss aversion, reflection effect and the coexistence of insurance and gambling. One notable exception that cannot be accommodated by the model is the class of the Allais-type paradoxes; they inherently require violation of independence, a key axiom in EU theory.

Referefence-dependence was first introduced in economics by Markowitz (1952). Along with probability weighting, reference-dependence is adopted as one of the two main tenets of Prospect Theory (Kahneman and Tversky, 1979, Tversky and Kahneman, 1992, Schmidt et al., 2008), the most popular alternative to the standard EU model. Since Prospect Theory accommodates most of the known EU paradoxes, including the Allais-type paradoxes, what is then the advantage of the model presented in this paper? The main advantage lies in uncovering sources of violations. Probability weighting in any form departs from EU theory as it necessarily involves violation of Independence. Reference-dependence changes interpretation as compared to consequentialism, but does not depart from EU theory. One contribution of this paper lies in showing that most of the evidence against the standard model (except for the Allais-type paradoxes) is in fact evidence against consequentialism and not against EU theory. Moreover, by providing exact equivalence results, one can identify sources of violation within the proposed reference-dependent EU model. For example, it is shown that WTA-WTP disparity arises in this model solely from loss aversion.

The model presented in this paper is closely based on the Reference-Dependent Subjective Expected Utility (RDSEU) model of Sugden (2003). Unlike previous reference-dependent models, the RDSEU model allows for a random reference point. Based on this model Schmidt et al. (2008) proposed Third-Generation Prospect Theory (TGPT). It assumes that gains and losses are nominal differences between consequences in a given act and a reference act in each state of the world. Under this assumption, TGPT generalizes the RDSEU model by adding rank-dependent weighting of state probabilities. Hence many properties of the model in this paper are inherited by a more general TGPT version.

The model proposed in this paper departs from RDSEU (or TGPT) approach in one important aspect: it restricts attention to objective uncertainty (exogenously given probabilites), thus simplifying the underlying formal model considerably. The advantage of this assumption lies in providing a more parsimonious formal representation, which is both quite natural and easier to manipulate with.

This paper is organized as follows. Section 3 introduces the formal model and some of its implications. Section 4 demonstrates how the model accommodates some of the well-known experimental evidence contradicting the standard EU model. Section 5 discusses the model and concludes.

## 2 Main assumptions

As mentioned before a model that is to be applied to data consists of two elements: a formal theory based on axioms, and an economic interpretation, which bridges the gap between the theory and the real world. These two elements are sometimes heavily intertwined with each other. This is the case here and therefore the formal model needs to be discussed along with the economic interpretation.

The decisions that will be analyzed in this paper are that under objective uncertainty (also referred to as risk). Thus probabilities are objectively given and since I want to stick to the vNM axioms (EU theory), it implies that the decision maker cares only about probability distributions over outcomes. This is true, but the question remains what are the outcomes and the associated probability distributions over outcomes. This question touches the issue of interpretation.

The interpretation I want to impose is broadly that of reference-dependence. But in any reference-dependent model the following issues need to be answered:

1. How are gains and losses defined?
    (a) How is reference defined in a given choice situation?
    (b) How are gains and losses defined given the reference?
2. How should gains and losses be evaluated:
    (a) across themselves?
    (b) among themselves?

I will answer these questions step-by-step, introducing the main assumptions of the model.

Reference dependence is modeled as a special case of the RDSEU model of Sugden (2003). This model is given in the context of subjective uncertainty. It is assumed that acts are defined on final wealth levels, and a given act $f$ is evaluated relative to a reference act $h$ by comparing consequences in the two acts state-wise. This state-contingent notion of reference-dependence implies that the DM takes into account stochastic dependence between $f$ and $h$. Furthermore, it is assumed that a reference act is allowed to be non-degenerate. This feature is also present in TGPT, but was neither present in the original version of prospect theory nor in the later cumulative version. It is furthermore assumed that the reference act is a *status quo* wealth. By making a specific assumption regarding a reference act, the number of degrees of freedom is reduced, thus making the model more testable. On the other hand, the *status quo* wealth is probably the most natural assumption to be made. Moreover, under this assumption the model accommodates the well-known EU paradoxes as will be shown in Section 4. I consider departures from this assumption only in Section 6. The assumption of a (possibly random) *status quo* wealth being the reference act makes it possible to model all possible kinds of exchanges. One example is that of selling a risky prospect, which is present in the selling price (or Willingness-To-Accept) elicitation task. Another example is exchanging one risky prospect for another; one important caveat in this case is that it requires the joint distribution of the two prospects to be specified – strength and direction of dependence between prospects play an important role in assessing profits from the exchange. These examples are analyzed in Section 3.1.

The assumption of a *status quo* reference point provides a clear answer to question 1(a) stated above. I now turn to question 1(b). It is assumed that gains and losses are defined by taking nominal differences between consequences in a given act $f$ and a reference act $h$ in each state. In this way a new act $l$ defined on wealth changes relative to the *status quo* wealth is formed; we refer to it as a prospect. This assumption is the same as the one made in TGPT. This assumption abstracts from possible wealth effects as it ignores what are the reference act consequences in absolute terms. It can be viewed as *a satisfactory approximation* (Kahneman and Tversky, 1979, pp. 277–278). The implications of this assumption are analyzed in Section 3.3. Since probabilities are assumed to be objective, as soon as a prospect is defined, the state-space needed to define it becomes redundant, as all that matters to the EU decision maker is probability distribution over ouctomes (or the induced probability distribution of prospect $l$). Thus a simplified formulation, which is referred to as the EU of wealth changes model, is used that dispenses with state space. Both formulations, the one that uses the state space and is needed to construct reference-dependent prospect and the one which takes prospects as given and evaluates them using the EU model, are introduced and examined formally in Section 3.

I now turn to question 2. Question 2(a) touches on a very important issue in any reference-dependent model, namely how the decision maker compares losses with equivalent gains. It is a very robust empirical finding that people usually exhibit loss aversion, meaning roughly that people generally dislike equal chances to either win \$$x$ or lose the same amount, no matter what this amount is. *We think it is clear that loss aversion and the tendency to isolate each risky choice must both be key components of a good descriptive theory of risk attitudes.* (Rabin and Thaler, 2001, p.230) Loss aversion is also a natural thing to expect. It is generally the case that richer people have more options to choose from. Hence if I gain money I expand my possibilities; if I lose money, my possibilities shrink (in real-life it may even lead to bankruptcy or prison). Even though it is not explicitly assumed, I will focus on preferences exhibiting loss aversion. Within these preferences I will focus on several classes of the utility functions with differing risk-attitudes. Instead of assuming theses functional form right away, they will arise as implications of more basic behavioral traits analyzed in Section 4.

## 3 Formal model

I now introduce the setup of Sugden (2003). Let $X \subseteq \mathbb{R}$ be the set of consequences, $S$ finite set of states. All subsets of $S$, called events, are allowed. A (Savage) act $f : S \to X$ is a random variable with finite support. The set of all acts is denoted by $F$. A degenerate act is an act where $f(s) = x$, for all $s \in S$, for some $x \in X$. The set of all degenerate acts is denoted by $F^d$. Depending on the context $x$ denotes either a degenerate act or an element of $X$. Probability measure on the full algebra on $S$ is specified by a probability function $\pi : S \to [0, 1]$, such that $\sum_{s\in S} \pi(s) = 1$.

I assume that preferences over acts are reference-dependent i.e. for all $f, g, h \in F$, $f \succsim_h g$ denotes $f$ is weakly preferred to $g$ viewed from $h$, the reference act. Sugden (2003) proves the following representation result referred to as the RDSEU model:

**Theorem (RDSEU).** *A reference-dependent preference relation $\succsim_r \subseteq F \times F \times F$ satisfies his eight axioms if and only if there exists a unique probability function $\pi$, and a relative value function $v : X \times X \to \mathbb{R}$ that is strictly increasing in the first argument and satisfies $v(x, y) = 0$, whenever $x = y$, is unique up to a positive linear transformation (i.e. multiplication by a positive constant), such that for all $f, g, h \in F$ the following holds:*

$$f \succsim_h g \iff \sum_{s\in S} v(f(s), h(s))\pi(s) \geq \sum_{s\in S} v(g(s), h(s))\pi(s) \tag{1}$$

As stated in Section 2 the model analyzed here makes a number of assumptions. Probabilities are exogenously given, $X$ represents final wealth levels and gains/losses in each state are defined as nominal differences in outcomes between a given act and a reference act. These assumptions make it possible to simplify the above model.

Let $\Delta X$ denote the set of all possible differences in wealth levels $x - y$, where $x, y \in X$. A prospect in the context of subjective uncertainty is an act $l : S \to \Delta X$. The set of all prospects is $\Delta F$. A typical element of the set $\Delta F$ will be denoted by $l := (x_1, E_1; ...; x_n, E_n)$, where $E_i$ are the collections of states for which $l(s) = x_i$. The acts $f, g, h$ are defined on final wealth levels. In specific choice situations it will be convenient to assume that an act $f$ defined on the set of final wealth levels $X$ can be decomposed into riskless initial wealth $W \in F^d$ and an additional uncertain prospect $l \in \Delta F$.

Alternatively, in the context of objective uncertainty, I will refer to prospects as probability distribution on $\Delta X$. The set of all such prospects is defined as:

$$L^{\Delta X} := \{p_l : \Delta X \to [0, 1],\ |\text{supp}(p_l)| < \infty,\ \sum_{x \in \text{supp}(p_l)} p_l(x) = 1\}$$

A typical element of $L^{\Delta X}$ will be denoted as $(x_1, p_1; ...; x_n, p_n)$, where $p_i = p_l(x_i)$, $p_l$ is the induced probability distribution of $l$. Depending on the context a prospect will either be an element of $\Delta F$ or an element from $L^{\Delta X}$.

Define a vNM preference relation $\succsim \subseteq L^{\Delta X} \times L^{\Delta X}$. The following is a well-known Expected Utility Theorem of von Neumann and Morgenstern (1944):

**Theorem (EU).** *A preference relation $\succsim \subseteq L^{\Delta X} \times L^{\Delta X}$ satisfies the vNM axioms if and only if there exists function $u : \Delta X \to \mathbb{R}$, unique up to positive affine transformation (i.e. up to addition of a constant and multiplication by a positive constant), such that for any $p_l, p_{l'} \in L^{\Delta X}$, the following holds:*

$$p_l \succsim p_{l'} \iff \sum_{x \in supp(p_l)} u(x) p_l(x) \geq \sum_{x \in supp(p_{l'})} u(x) p_{l'}(x) \qquad (2)$$

The proposition below demonstrates that under the assumptions made, one can simplify the model in (1) by replacing it with the one in (2).

**Proposition 3.1.** *The following two representations are equivalent:*

1. *The RDSEU representation given by (1) with the value function $v(x, y) = u(x - y)$ for all $x, y \in X$, where $u : \Delta X \to \mathbb{R}$ is a strictly increasing, continuous function satisfying $u(0) = 0$.*

2. *the vNM representation given by (2), where $l, l' : S \to \Delta X$, $l(s) = f(s) - h(s)$, $l'(s) = g(s) - h(s)$, for all $s \in S$ and $p_l, p_{l'} \in L^{DeltaX}$, s.t. $p_l(x) = \sum_{s:l(s)=x} \pi(s)$, $p_{l'}(x) = \sum_{s:l'(s)=x} \pi(s)$, for $x \in \Delta X$.*

*Proof.* The axioms of Sugden (2003) applied in the context of objective uncertainty (risk) imply the vNM axioms. It remains to be shown that there is a translation of objects in one representation into the corresponding objects in another representation. That this is the case, can easily be checked. We refer to $p_l, p_{l'}$ as to the induced probability distributions of $l$ and $l'$, respectively. Note that $p_l(x) \geq 0$, for all $x \in \Delta X$ and $\sum_{x \in \Delta X} p_l(x) = 1$. □

### 3.1 Choice and pricing in the EU of wealth changes model

Assume that the reference act $h$ is current wealth level. Table 1 presents several typical choice problems and the way they are handled by the model.

| Problem description | $h$ | $f$ | $g$ | Interpretation |
|---|---|---|---|---|
| a) Accept or reject prospect $l$ | $W$ | $W+l$ | $W$ | $f$: accept $l$, $g$: reject $l$ |
| b) Choose btw. prospect $l_1$ and $l_2$ | $W$ | $W+l_1$ | $W+l_2$ | $f$: accept $l_1$, $g$: accept $l_2$ |
| c) Exchange prospect $l_1$ for $l_2$ | $W+l_1$ | $W+l_2$ | $W+l_1$ | $f$: exchange, $g$: do not exchange |
| d) Buying price $B$ elicitation | $W$ | $W+l-B$ | $W$ | $f$: buy prospect $l$, $g$: do not buy |
| e) Selling price $S$ elicitation | $W+l$ | $W+S$ | $W+l$ | $f$: sell prospect $l$, $g$: do not sell |

Table 1: Typical choice situations in the wealth-changes model.

By proposition 3.1, such choice and valuation decisions can be represented by the following conditions:

a) Accept prospect $l$: $\mathbb{E}u(l) \geq 0$

b) Choose prospect $l_1$ over prospect $l_2$: $\mathbb{E}u(l_1) \geq \mathbb{E}u(l_2)$

c) Exchange prospect $l_1$ for $l_2$: $\mathbb{E}u(l_2 - l_1) \geq 0$

d) Buying price elicitation: $\mathbb{E}u(l - B) = 0$

e) Selling price elicitation: $\mathbb{E}u(S - l) = 0$

Buying price $B \equiv \text{WTP}(l)$, also known as Willingness-To-Pay, is the maximal sure amount the decision maker is willing to pay for the right to prospect

$l$. Selling price $S \equiv \mathrm{WTA}(l)$, also known as Willingness-To-Accept, is the minimal sure amount the decision maker is willing to accept to forego the rights to prospect $l$. These two concepts were introduced by Luce and Raiffa (1957).

### Connection to regret theory

Note that the reference act $h$ has to be the same when evaluating acts $f$ and $g$. Thus it is impossible in the current setup to analyze the reference act that is dependent on the act being evaluated. One important example of such dependence is a simplified model of regret Bell, 1982,Loomes and Sugden, 1982). Regret of choosing alternative $a_1$ over $a_2$ is measured in general as $v(u(a_1)-u(a_2))$, where $v$ is the regret function and $u(a)$ the utility of a consequence $a$. Suppose, however, that the formula is simplified to $v(a_1-a_2)$, and $v=u$. In this case one can think of the regret model as a reference-dependent model with gains and losses defined the same way as in the EU of wealth changes model with one important caveat: whenever acts $f$ and $g$ are compared, $g$ is a reference act for $f$, and $f$ is a reference act for $g$. This motivates the definition of the regret-balanced measure of a prospect. Suppose that $C$ is a degenerate prospect, $h=W$ a degenerate act and $l$ any prospect. Let $f=W+l-C$, $g=W-l+C$. Then $C$ is called a regret-balanced measure of $l$ if it satisfies the following:

$$\mathbb{E}u(l-C)=\mathbb{E}u(C-l)$$

One can think of it in the following way: I can either accept prospect $l$ or take a sure amount of money $C$. If I accept the prospect, the foregone sure amount $C$ is my reference. If I take the amount $C$, the foregone prospect $l$ is my reference. The amount $C$ can be interpreted as the amount for which the expected regret when I choose prospect $l$ is equal to the expected regret when I choose the sure amount $C$. Thus $C$ is a regret-balanced measure of $l$. Even though the regret-balanced measure of $l$ cannot be defined in the current model, because it would require a different preference foundation, it is treated as a useful benchmark and the way to compare the regret and the reference-dependent approaches.

## 3.2 Loss aversion utility functions

In this section I introduce a couple of utility functions classes that exhibit loss aversion. These classes will later arise in results concerning EU paradoxes. The general definition of loss aversion (Kahneman and Tversky, 1979) is the following:

**Definition 1.** *The decision maker exhibits loss aversion if the utility function* $u(\cdot)$ *representing his preferences satisfies:* $u(x) < -u(-x)$, *for all* $x \in \Delta X \setminus \{0\}$.

**Lemma 1.** $u(x) < -u(-x)$ *holds for* $x > 0$ *if and only if the same holds for* $x < 0$.

*Proof.* Let $x > 0$ and $y = -x$. Then $u(x) < -u(-x)$ if and only if $u(-y) < -u(y)$ or $u(y) < -u(-y)$. Since $x$ was an arbitrary positive number the proof is finished. $\square$

The following are possible functional forms of the reference-dependent utility function $u : \Delta X \to \mathbb{R}$ that will arise in the results of Section 4. From now on I assume that $x \in \Delta X$ and the loss aversion parameter $\lambda$, whenever present, is strictly positive.

A1. **General reflected** utility function:

$$u(x) = \begin{cases} \bar{u}(x) & \text{if } x \geq 0, \\ -\lambda \bar{u}(-x) & \text{if } x < 0. \end{cases} \tag{3}$$

A function $\bar{u}(\cdot)$ is continuous and strictly increasing with $\bar{u}(0) = 0$.

A2. **Reflected homogeneous** utility function:

$$u(x) = \begin{cases} x^{\alpha} & x \geq 0, \\ -\lambda(-x)^{\alpha} & x < 0, \end{cases} \quad \text{where } \alpha > 0 \tag{4}$$

A3. **Concave** utility function: $u$ is strictly increasing and concave over $\Delta X$ and $u(0) = 0$

A4. **Simple loss aversion** utility function:

$$u(x) = \begin{cases} x & \text{if } x \geq 0 \\ \lambda x & \text{if } x < 0 \end{cases} \tag{5}$$

The exemplary graphs of these functions are provided in Figure 3.2.

## 3.3 Gains and losses – implications of the definition

I now proceed to discussing implications of the way gains and losses are defined in the model. The following propositions establish three invariance results. The first one determines the type of reference changes under which arbitrary preferences remain invariant in the model.

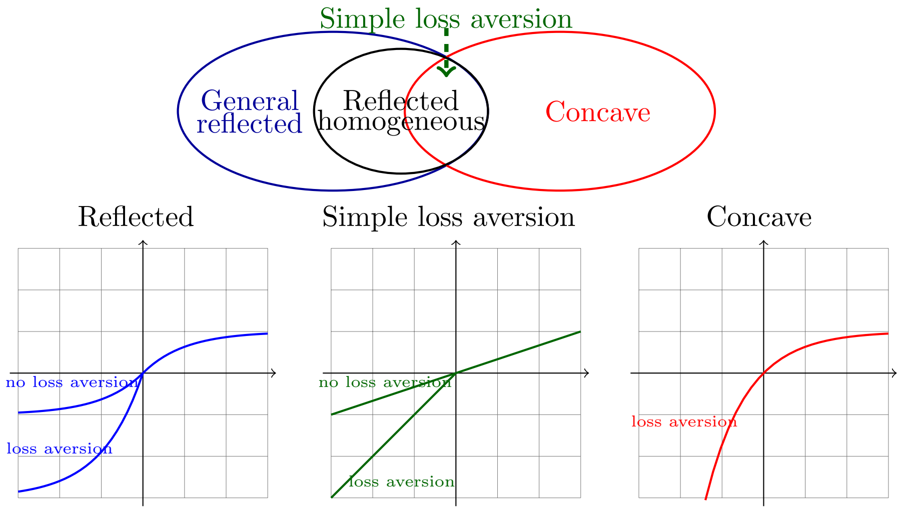


Figure 1: Different classes of utility functions.

**Proposition 3.2.** *For any acts $f, g, h, h' \in F$, the following holds: $f \succsim_h g$ if and only if $f + h' - h \succsim_{h'} g + h' - h$.*

*Proof.* Follows from the fact that the utility function is defined on the difference between two acts – when this difference stays unchanged, the preference is also unchanged. □

The above result can be translated easily into practical implications. For example consider the selling price elicitation task. Suppose that instead of $h = W + l$ a reference act is set to be $h' = W$. Then, in order to obtain the same selling price one needs to set $f' = W - l + S$ instead of $f = W + S$, and $g' = W$ instead of $g = W + l$. This implies that the following two elicitation procedures are equivalent in the EU of wealth changes model:

a) The decision maker owns the right to play prospect $l$. He is asked what is the minimal sure amount for which he would be willing to forego this right.

b) The decision maker does not own the right to play prospect $l$. He is asked what is the minimal sure amount for which he would be willing to take a short position on this prospect $l$.

The other two invariance results presented in this section take the opposite approach: they determine preferences within the model that are invariant to changing a reference act: one concerns an arbitrary reference act and the other a nondegenerate reference act.

**Proposition 3.3.** *The following two conditions are equivalent:*

i. *For any two distinct degenerate acts* $h, h'$, $f \succsim_h g \iff f \succsim_{h'} g, \forall f, g \in F$.

ii. *The utility function* $u$ *exhibits CARA.*

*Proof.* Condition i. can be written in terms of the model as:

$$\mathbb{E}u(f') \geq \mathbb{E}u(g') \iff \mathbb{E}u(f' + h'') \geq \mathbb{E}u(g' + h'')$$

where $f' = f - h$, $g' = g - h$ are any acts and $h'' = h - h'$ is a nonzero degenerate act. By Pratt (1964) theorem this holds if and only if the utility function $u$ exhibits constant absolute risk aversion. □

Combined with Proposition 3.2, the above result implies that shifting acts being compared (i.e. $f$ and $g$) by the same constant and/or shifting the reference act by a (possibly different) constant does not change preferences if and only if $u$ belongs to a CARA class.

**Proposition 3.4.** *The following two conditions are equivalent:*

i. *For any two distinct acts* $h, h'$, $f \succsim_h g \iff f \succsim_{h'} g, \forall f, g \in F$.

ii. *The utility function* $u$ *exhibits risk neutrality.*

*Proof.* The utility function $u$ exhibits risk neutrality if and only if it is of the form $u(x) = ax + b$, where $a, b \in \mathbb{R}$, $a > 0$. If utility function is of this form obviously condition i. holds. For the opposite direction take any acts $f, g \in F$, and $h = 0$ $h' = f$. By condition i. $\mathbb{E}_f u(f) \geq \mathbb{E}_g u(g)$ is equivalent to $0 \geq \mathbb{E}_{f,g} u(g - f)$, where $\mathbb{E}_f, \mathbb{E}_g, \mathbb{E}_{f,g}$ denote the expectation's operator with respect to the distribution of $f$, of $g$ and teh joint distribution of $f$ and $g$, respectively. This in turn is equivalent to $\mathbb{E}_{f,g}[u(f-g) - u(f) + u(g)] = 0$. Since this holds for any $f, g \in F$, it must be equivalent to $u(x - y) = u(x) - u(y)$ for any $x, y \in \mathbb{R}$. This in turn is the famous Cauchy functional equation for which the only solution is $u(x) = ax + b$, where $x \in \mathbb{R}$, $a, b \in \mathbb{R}$. Since $u$ is strictly increasing we have $a > 0$. □

The above proposition clarifies that within the model invariance towards an arbitrary act is equivalent to risk neutrality. It implies that except for risk-neutral preferences, reference dependence is always present if the reference wealth is allowed to be random.

### 3.4 Properties of WTA-WTP measures

We now state several properties of WTA and WTP which will be useful later on.

We begin by stating some general properties of WTA, WTP and the regret-balanced measure $C$.

**Property 1.** *Willingness-to-pay and willingness-to-accept for a prospect $l \in \Delta F$ with the support equal to $[\underline{x}, \overline{x}]$, $\underline{x}, \overline{x} \in \Delta X$, $\underline{x} < \overline{x}$ are in the interior of the support of this prospect, i.e.*

$$\text{WTA}(l),\ C(l),\ \text{WTP}(l) \in (\underline{x}, \overline{x})$$

*Proof.* Straightforward from definition. □

**Property 2.** *Given a prospect $l \in \Delta F$ and a constant $c \in \mathbb{R}$ such that $l + c \in \Delta F$ , the following holds:*

$$WTX(l + c) = WTX(l) + c \quad \text{where} \quad X \in \{A, P\}.$$

*Proof.* We prove it for WTP. Directly from definition:

$$\begin{aligned} 0 &= \mathbb{E}u(l - \text{WTP}(l)) \\ &= \mathbb{E}u(l + c - (\text{WTP}(l)) + c) \\ &= \mathbb{E}u((l + c) - \text{WTP}(l + c)) \end{aligned}$$

Similarly for WTA. □

This property tells us that shifting all the outcomes of a prospect by a constant should change the measures exactly by this constant.

**Property 3.** *Given prospects $l, -l \in \Delta F$, the following holds:*

$$\text{WTX}(l) = -\text{WTY}(-l) \quad \text{where} \quad \text{X}, \text{Y} \in \{\text{A}, \text{P}\} \quad \text{and} \quad \text{X} \neq \text{Y}.$$

*Proof.* We prove it for WTA($l$). Directly from definition:

$$\begin{aligned} 0 &= \mathbb{E}u(\text{WTA}(l) - l) \\ &= \mathbb{E}u(-l - (-\text{WTA}(l))) \\ &= \mathbb{E}u((-l) - \text{WTP}(-l)) \end{aligned}$$

Similarly for WTP($l$). □

Suppose that $l$ is a gain prospect, i.e. $\sum_{x \geq 0} p_l(x) = 1$. Then $-l$ is a loss prospects, i.e. $\sum_{x \leq 0} p_{-l}(x) = 1$. The above property implies that the negative of what I am willing to pay for a given gain equals what I am willing to accept for the loss of the same magnitude.

**Complementary symmetry**

Complementary symmetry was first analyzed by Birnbaum and Zimmermann (1998) and then by (Birnbaum et al., 2016).

Let $l := (y, p; x, 1-p)$ and $l' := (y, 1-p; x, p)$ be two prospects in $\Delta F$. We say that complementary symmetry holds if $\mathrm{WTP}(l) + \mathrm{WTA}(l') = x + y$.

**Proposition 3.5.** *Complementary symmetry holds in RDSEU for any utility function.*

*Proof.* Define $\theta(p) = -\frac{1-p}{p}$ and assume that $B = \mathrm{WTP}(l)$ and $S = \mathrm{WTA}(l')$. Then it is straightforward to show that:

$$\frac{u(y-B)}{u(x-B)} = \frac{u(S-x)}{u(S-y)} = \theta(p)$$

Suppose that complementary symmetry does not hold. There are two cases to consider:

a) $x + y > S + B$: then since $u$ is strictly increasing, the nominator on the LHS of the above expression is strictly higher than the nominator on the RHS. Equality will hold only if the same holds for the denominators, i.e. $B - x > y - S$, thus contradicting the assumption.

b) $x + y < S + B$: by a similar argument, we get contradiction as well.

Hence it must be that $x + y = B + S$. □

# 4 Paradoxes

## 4.1 WTA-WTP disparity and loss aversion

WTA-WTP disparity was first documented by Knetsch and Sinden (1984).

**Proposition 4.1.** *The decision maker whose preferences are represented by the utility function $u(\cdot)$ exhibits loss aversion if and only if given any nondegenerate prospect $l \in \Delta F$, the following holds:* $\mathrm{WTA}(l) > C(l) > \mathrm{WTP}(l)$.

*Proof.* ($\Rightarrow$): Denote $C \equiv C(l)$. Note that if $C = x$ then $u(x - C) = -u(C - x)$, and for $C \neq x$ $u(x - C) < -u(C - x)$ by loss aversion and Lemma 1. Since prospect $l$ is nondegenerate, there are consequences $x$ in the support of $l$, such that $C \neq x$. Hence we have:

$$2\mathbb{E}u(l - C) < \mathbb{E}u(l - C) - \mathbb{E}u(C - l) < -2\mathbb{E}u(C - l)$$

By definition of $C$ the middle term is equal to zero, so we have:

$$\mathbb{E}u(l - C) < 0 = \mathbb{E}u(l - \text{WTP}(l))$$
$$\mathbb{E}u(C - l) < 0 = \mathbb{E}u(\text{WTA}(l) - l)$$

By strict monotonicity of $u$ we thus obtain that $\text{WTA}(l) > C(l) > \text{WTP}(l)$.

($\Leftarrow$): This part will be proven by contradiction. Suppose that the decision maker does not exhibit loss aversion, i.e.

$$\exists x^* \neq 0: \ u(x^*) \geq -u(-x^*). \tag{6}$$

Given the same utility function $u$ construct prospect $l = (\overline{x}, p; \underline{x}, 1-p)$, such that $x^* = \overline{x} - C = C - \underline{x}$, where $C \equiv C(l)$. This implies that $C(l) = \frac{\overline{x}+\underline{x}}{2}$. It is always possible to construct such prospect since $u$ is strictly increasing and continuous. Using the definition of $\text{WTP}(l)$ and $\text{WTA}(l)$ and Lemma 1 we have:

$$2\mathbb{E}u(l - C) \geq \mathbb{E}u(l - C) - \mathbb{E}u(C - l) \geq -2\mathbb{E}u(C - l)$$

A similar argument as in the $\Rightarrow$ part shows then that $\text{WTP}(l) \geq C(l) \geq \text{WTA}(l)$, which contradicts the claim. □

The above proposition is quite general. I will focus on two special cases. Consider the utility function General Reflected (i.e. of the form specified in (3). This utility function exhibits loss aversion if an only if $\lambda > 0$.

**Corollary 1.** *For a nondegenerate prospect $l \in \Delta F$ and the general reflected utility function given by (3) the following holds:*

$$\lambda > 1 \iff \text{WTA}(l) > C(l) > \text{WTP}(l)$$

*Proof.* Directly from Proposition 4.1. □

The above corollary shows that in prospect theory, a willingness to accept/willingness to pay disparity may be explained solely by loss aversion.

Another special case of a loss aversion utility function is a Concave utility function, specified in A3. in Section 1. For this class of functions an even stronger condition holds:

**Proposition 4.2.** *For a nondegenerate prospect $l \in \Delta F$, given the utility function that is strictly increasing and bounded with $u(0) = 0$, the following holds:*

$$u(\cdot) \textit{ is strictly concave} \iff \text{WTA}(l) > \mathbb{E}[l] > \text{WTP}(l)$$

*Proof.* By strict Jensen's inequality:

$$\begin{aligned} 0 &= \mathbb{E}u(l - \text{WTP}(l)) < u(\mathbb{E}[l] - \text{WTP}(l)) \\ 0 &= \mathbb{E}u(\text{WTA}(l) - l) < u(\text{WTA}(l) - \mathbb{E}[l]) \end{aligned}$$

Since $u(0) = 0$ and $u$ is strictly increasing, the conclusion follows. □

One special case of the General Reflected utility function given by (3) which is simultaneously overall concave (but not strictly concave) is when there is only loss aversion and otherwise no additional curvature. This is the function of the form Simple Loss Aversion given in (5). The following corollary to Proposition 4.2 applies to this class of utility functions:

**Corollary 2.** *For a nondegenerate prospect $l \in \Delta F$ and the simple loss aversion utility function given by (5) the following holds:*

$$\lambda > 1 \iff \text{WTA}(l) > \mathbb{E}[l] > \text{WTP}(l)$$

*Proof.* Proposition 4.2 concerns a strictly concave utility function, whereas this utility function is only weakly concave. It is however straightforward to observe that in the WTA/WTP calculation a kinked area of the utility function (5) must be used and hence the conclusion must be the same. □

The conclusion of this section is that the gap between WTA and WTP in the case of preferences defined over wealth changes may be explained by loss aversion. This requirement defines a wide class of utility functions and in particular, an S shaped utility function with loss aversion as well as a traditional overall concave utility function over the whole real line.

## 4.2 Preference homogeneity

Tversky and Kahneman (1992) postulate that the CPT preferences are homogeneous, which means that multiplying the outcomes of a prospect $l$ by a constant $k > 0$ multiplies its cash equivalent by the same constant. They claim (but without providing a proof) that preference homogeneity in the CPT model is both necessary and sufficient to represent their value function as:

$$u(x) = \begin{cases} x^{\alpha} & x \geq 0 \\ -\lambda(-x)^{\beta} & x < 0. \end{cases} \qquad (7)$$

where $\alpha, \beta > 0$.

In what follows we assume that the cash equivalent of a prospect is defined as the Willingness-To-Accept of this prospect, but a similar result holds for other possible definitions of a cash equivalent. The following proposition establishes that in the RDSEU model (and in the CPT as well) preference homogeneity holds if and only if the utility function takes the form of (4), i.e. the special case of the function (7) where $\alpha$ equals $\beta$, such that there is the same level of curvature (but with the opposite sign) for gains and for losses. This result thus casts doubts on the claim made by Tversky and Kahneman (1992).

**Definition 2.** *Preference homogeneity holds if and only if for any prospect* $l \in \Delta F$*,* $\text{WTA}(kl) = k\text{WTA}(l)$*, for all* $k > 0$*.*

**Proposition 4.3.** *Preference homogeneity holds if and only if the utility function is reflected homogeneous given by (4).*

*Proof.* It is easy to verify that if the utility function is given by 4, then preference homogeneity holds. Let us focus now on the opposite direction. For any prospect $l \in \Delta F$ and scalar $k > 0$, the Willingness-To-Accept of prospect $kl$ is defined as follows:

$$\mathbb{E}u(WTA(kl) - kl) = 0 \tag{8}$$

Preference homogeneity holds if and only if $u$ is a strictly increasing and continuous function satisfying:

$$u(kx) = \phi(k)u(x), \quad k > 0,\ x, kx \in \mathbb{R} \tag{9}$$

where $\phi$ is also a strictly increasing and continuous function defined on $(0, \infty)$. In what follows we will solve this functional equation which belongs to the class of Pexider equations.

When $x = 0$, it follows from the equation that $u(0) = 0$. When $k = 1$, we obtain $\phi(1) = 1$, and when $x = 1$, we obtain $u(k) = u(1)\phi(k)$, for $k > 0$. Define $a := u(1)$ and $t(x) = u(x)/a$, for $x \in \mathbb{R}$. We can now rewrite (9) as:

$$t(kx) = t(k)t(x), \quad k > 0,\ x, \in \mathbb{R} \tag{10}$$

We will now consider two cases.

a) First suppose that $x > 0$. Define $k' = \log_b(k)$, $x' = \log_b(x)$, where $x, k, b > 0$ and take logarithms base $b$ on both sides of equation (10). We obtain:

$$\log_b t\left(b^{k'+x'}\right) = \log_b t\left(b^{k'}\right) + \log_b t\left(b^{x'}\right), \quad k', x' \in \mathbb{R}$$

Define $s(x) = \log_b t\left(b^{x}\right)$, for $x \in \mathbb{R}$. The equation above becomes:

$$s(k' + x') = s(k') + s(x'), \quad k', x' \in \mathbb{R}$$

This is Cauchy equation and the only continuous and strictly increasing function satisfying it is $s(x) = \alpha x$, $\alpha > 0$, $x \in \mathbb{R}$. Using the definition of $s$ and substituting $x = b^{x'}$ we obtain $s(x') = \log_b t\left(b^{x'}\right) = \alpha x'$, $x' \in \mathbb{R}$ or $t(x) = x^{\alpha}$, for $x > 0$. Finally we substitute back $t(x) = \phi(x) = u(x)/a$ to get:

$$u(x) = ax^{\alpha}, \quad x > 0$$
$$\phi(k) = k^{\alpha}, \quad k > 0.$$

b) Now suppose that $x < 0$. Define $k' = \log_c(k)$, $x' = \log_c(-x)$, where $c, k, -x > 0$ and take logarithms base $c$ on the negative of both sides of equation (10) to obtain:

$$\log_c\left[-t\left(c^{k'+x'}\right)\right] = \log_c t\left(c^{k'}\right) + \log_c\left[-t\left(c^{x'}\right)\right], \quad k', x' \in \mathbb{R}$$

Define $s(x) = \log_c\left[-t\left(c^{x}\right)\right]$ and $r(x) = \log_c t\left(c^{x}\right)$ and subtitute back into the equation:

$$s(k' + x') = r(k') + s(x'), \quad k', x' \in \mathbb{R}$$

This is one of the standard Pexider equations and the only continuous and strictly increasing functions that satisfy this equation are:

$$s(x) = \beta x + \gamma$$
$$r(x) = \beta x$$

where $\beta > 0$, and $\gamma \in \mathbb{R}$. Using the definition of $s$ and substituting $-x = c^{x'}$, we obtain $\log_c\left[-t\left(-c^{x'}\right)\right] = \beta x' + \gamma$, for $x' \in \mathbb{R}$ or $t(x) = -(-x)^{\beta} c^{\gamma}$, for $x < 0$. Similarly, using the definition of $r$ and substituting $k = c^{k'}$, we obtain $\log_c\left[t(c^{k'})\right] = \beta k'$, for $k' \in \mathbb{R}$ or $t(k) = k^{\beta}$, for $k > 0$. Finally we substitute back $t(x) = \phi(x) = u(x)/a$, to obtain:

$$u(x) = -ac^{\gamma}(-x)^{\beta}, \quad x < 0$$
$$\phi(k) = k^{\beta}, \quad k > 0.$$

Now to ensure preference homogeneity the exponents in functions *phi* in both cases a) and b) have to be equal. which means that $\alpha = \beta$. Let us also normalize the utility function by dividing the utility for gains and the utility for losses by a common number $a$. In order to ensure continuity of $u$ defined over the whole real line we need to have $u(0) = 0$. Define $\lambda = c^{\gamma}$ and we finally conclude that the utility function takes the form specified by (4). □

### 4.3 Preference reversal for concave utility functions

Preference reversal was first analyzed by psychologists ( Lichtenstein and Slovic, 1971, Lindman, 1971 and Lichtenstein and Slovic, 1973). Later it was further investigated by economists (Grether and Plott, 1979).

First, recall that traditional preference reversal is not possible within expected utility when preferences are defined over wealth levels, irrespective of whether these wealth levels are interpreted narrowly as levels of gambling wealth, for instance, or whether they are interpreted traditionally as total wealth levels. On the other hand, when preferences are defined over wealth changes, it turns out that traditional preference reversal is possible. Schmidt et al. (2008) shows that preference reversal may occur in third generation prospect theory. They calibrate for which values of parameters a version of parametrized prospect theory is compatible with preference reversal. Below I will demonstrate how preference reversal may be obtained as a generic element.

Let $l_P \equiv (x, p; 0, 1-p)$ and $l_\$ \equiv (y, q; 0, 1-q)$ be two binary prospects in $\Delta F$ such that $y > x > 0$ and $1 > p > q > 0$. prospect $l_P$ will be called the P-bet and prospect $l_\$$ will be called the \$-bet. In what follows I want to demonstrate that preference reversal is possible under quite general assumptions. For that I need two lemmas:

**Lemma 2** (Three strings lemma). *Utility function $u(\cdot)$ is strictly concave if and only if for $a > b > c$ the following holds:*

$$\frac{u(a)}{a} < \frac{u(b)}{b} < \frac{u(c)}{c}. \tag{11}$$

*Proof.* The result is standard. □

TODO: Try to relax: strict concavity. We need a much weaker condition

**Lemma 3.** *Suppose that $px = qy$. Given a utility function $u : \mathbb{R} \to \mathbb{R}$ that is strictly increasing, continuous and bounded with $u(0) = 0$, the following holds:*

$$u(\cdot) \textit{ is strictly concave} \iff \mathrm{WTA}(l_\$) > \mathrm{WTA}(l_P)$$

*Proof.* Define $S \equiv \mathrm{WTA}(l_P)$ to save on notation. $S$ satisfies the following equation:

$$pu(S - x) + (1 - p)u(S) = 0 \tag{12}$$

Using the three-strings lemma $u(\cdot)$ is strictly concave if and only if

$$\begin{aligned} qu(S-y)+(1-q)u(S) &< q\frac{S-y}{S-x}u(S-x)+(1-q)u(S) \\ &= u(S-x)\left[q\frac{S-y}{S-x}-p\right]+(p-q)u(S) \\ &= u(S-x)\frac{qS-qy-pS+px}{S-x}+(p-q)u(S) \\ &= S(p-q)\left[\frac{u(S)}{S}-\frac{u(S-x)}{S-x}\right] \\ &< 0 \end{aligned}$$

where both inequalities follow from (11), the first equality from (12) and the third equality from the fact that $px = qy$. □

We now state the proposition.

**Proposition 4.4.** *Suppose that $px = qy$. Preference reversal occurs if $u(\cdot)$ is strictly concave.*

*Proof.* Suppose that $u(\cdot)$ is strictly concave. Then $\text{WTA}(l_\$) > \text{WTA}(l_P)$, by Lemma 3. Since $x < y$, $\frac{u(x)}{x} > \frac{u(y)}{y}$, by the three strings lemma for a concave function $u(\cdot)$. Hence the following holds:

$$\mathbb{E}u(l_P) = pu(x) > p\frac{x}{y}u(y) = qu(y) = \mathbb{E}u(l_\$)$$

So prospect $l_P$ or a P-bet is chosen over prospect $l_\$$ or a \$-bet in a direct choice and yet $\text{WTA}(l_\$) > \text{WTA}(l_P)$ as required. □

As a direct corollary to Proposition 4.4 we have the following result:

**Corollary 3.** *For a simple loss aversion utility function given by 5, preference reversal is possible if and only if $\lambda > 1$*

*Proof.* By Lemma 3 if If $px = qy$, so that the decision maker is indifferent between prospect $l_P$ and $l_\$$ in a direct choice, then

$$\lambda > 1 \iff \text{WTA}(l_\$) > \text{WTA}(l_P)$$

The rest follows directly from the proof of Proposition 4.4. □

Concavity of a utility function is sufficient for preference reversal in the above example. However it is not necessary. In particular, Schmidt et al. (2008) show that preference reversal is possible with an S-shaped prospect utility function, which is convex for losses. If one wants to obtain a possibility of preference reversal for specific prospects and not as a generic feature of the model, then the following requirement, which is weaker than the overall concavity of the utility function, may be imposed: For a given P-bet and a given \$-bet and utility function $u(\cdot)$, define $S = \mathrm{WTA}(l_P)$. Then:

$$\frac{u(S-y)}{S-y} > \frac{u(S-x)}{S-x} > \frac{u(S)}{S}$$

### 4.4 Reflection effect and loss aversion

Reflection effect was first analyzed by Kahneman and Tversky (1979).

**Definition 3.** *Strong reflection holds if for any three acts $f, g, h \in F$, such that $f(s) - h(s) \geq 0$, $g(s) - h(s) \geq 0$, for all $s \in S$ the following holds:*

$$f \succsim_h g \iff -g \succsim_{-h} -f$$

In what follows we assume that if $l \in \Delta F$ then $-l \in \Delta F$. Given $f, h \in F$, a prospect $l \in \Delta F$, such that $l(s) = f(s) - h(s)$, for $s \in S$ shall be denoted by $f - h$. Such a prospect is called a gain prospect if for all $s \in S$, $l(s) \geq 0$. It is called a loss prospect if for all $s \in S$, $l(s) \leq 0$. If prospect $l$ is neither a loss prospect nor a gain prospect it is a mixed prospect. The associated probability distributions, will be called, the gain prospect, the loss prospect and the mixed prospect, respectively.

**Proposition 4.5.** *Strong reflection holds if and only if $u$ is of the general reflected form:*

$$u(x) = \begin{cases} \bar{u}(x), & x \geq 0 \\ -\lambda\bar{u}(-x), & x < 0 \end{cases} \tag{13}$$

*where $\bar{u}$ is strictly increasing and continuous with $\bar{u}(0) = 0$ and $\lambda > 0$.*

*Proof.* Let $f, g, h \in F$ be as in the definition of the strong reflection effect. Let's define prospects $l, l', -l, -l' \in \Delta$, such that $l := f - h$, $l' := g - h$, $-l := h - f$, $-l' := h - g$ and their associated probability distributions $p_l$ and $p_{l'}$, $p_{-l}$, $p_{-l'}$. It follows that $l, l'$ are gain prospects and $-l, -l'$ are loss prospects. We assume that there is a best and a worst gain prospect $l_b$, $l_w$, respectively, and that the best and worst prospects among loss prospects are $-l_w$ and $-l_b$, respectively. For convenience we assume that $l_w = 0$, such that we can immediately get $u(0) = 0$. To simplify notation we shall denote $p := p_l$, $q := p_{l'}$, $-p := p_{-l}$, $-q := p_{-l'}$ and $p^b := p_{l_b}$, $p^w := p_{l_w}$, $-p^b := p_{-l_b}$ and $-p^w := p_{-l_w}$.

($\Leftarrow$): Let the utility function $u$ be of the form specified by (3). Then

$$\begin{aligned} p \succsim q &\iff \sum_{x\in X} p(x)\bar{u}(x) \geq \sum_{x \in X} q(x)\bar{u}(x) \\ &\iff -\sum_{x\in X} q(x)\lambda\bar{u}(-(-x)) \geq -\sum_{x\in X} p(x)\lambda\bar{u}(-(-x)) \\ &\iff -q \succsim -p \end{aligned}$$

($\Rightarrow$): Let $u^g : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing and continuous function with $u^g(0) = 0$ that represents the decision maker's preferences for gain prospects. Let $\alpha_p \in [0,1]$ be such that: $p \sim \alpha_p p^b + (1-\alpha_p)p^w$. Since $u^g$ is vNM utility, it satisfies the EU property: $u^g(p) = \alpha_p u^g(p^b) + (1-\alpha_p)u^g(p^w)$ so that $\alpha_p = \frac{u^g(p)-u^g(p^w)}{u^g(p^b)-u^g(p^w)}$. Similarly define $\alpha_q$ for prospect $q$. Let $u^l : \mathbb{R}^- \to \mathbb{R}^-$ be a utility function for losses. Since it is also a vNM utility function, there exists $\beta_{-p} \in [0,1]$ such that: $-p \sim \beta_{-p}(-p^w) + (1-\beta_{-p})(-p^b)$. Similarly define $\beta_{-q}$ for prospect $-q$. Using strong reflection and EU, we have:

$$\alpha_p \geq \alpha_q \iff p \succsim q \iff -q \succsim -p \iff \beta_{-q} \geq \beta_{-p}$$

Hence it must be that $\beta_{-p} = 1-\alpha_p$ and $\beta_{-q} = 1-\alpha_q$. So:

$$\begin{aligned} u^l(-p) &= \beta_{-p}u^l(-p^w) + (1-\beta_{-p})u^l(-p^b) \\ &= (1-\alpha_p)u^l(-p^w) + \alpha_p u^l(-p^b) \\ &= (1-\alpha_p)[u^l(-p^w) - u^l(-p^b)] + u^l(-p^b) \\ &= \frac{u^g(p)-u^g(p^w)}{u^g(p^b)-u^g(p^w)}[u^l(-p^w) - u^l(-p^b)] + u^l(-p^b) \\ &= -\tfrac{u^l(-p^w)-u^l(-p^b)}{u^g(p^b)-u^g(p^w)}u^g(p) + u^l(-p^b) + u^g(p^b)\frac{u^l(-p^w)-u^l(-p^b)}{u^g(p^b)-u^g(p^w)} \end{aligned}$$

It follows that we can write $u^l(-p) = -\lambda u^g(p) + \theta$, where $\lambda, \theta \in \mathbb{R}$, $\lambda > 0$. Since we require that the overall utility function $u$ is continuous and satisfies $u(0) = 0$ we must have $\theta = 0$ and the proof is finished. $\square$

Note that the above result combined with loss aversion gives the standard prospect theory utility function as in (3) with $\lambda > 1$.

**Proposition 4.6.** *Loss aversion holds if and only if for all prospects $l \neq 0$ such that $l = -l$, where $l \in \Delta F$, the following holds: $\mathbb{E}u(l) < 0$.*

*Proof.* ($\Leftarrow$): Let $l := (x_1, p_1; ...; x_n, p_n)$, such that $x_1 < ... < x_i < 0 \leq x_{i+1} < ... < x_n$, $l \in \Delta F$. Since $l = -l$, $x_k = -x_{n+1-k}$, $p_k = p_{n+1-k}$, for $k = 1, 2, ..., i$.

If $x_{i+1} > 0$, then $n = 2i$. If $x_{i+1} = 0$, then $n = 2i+1$. By loss aversion we have:

$$\begin{aligned}\mathbb{E}u(l) &= \sum_{k\leq i} p_k u(x_k) + \sum_{k\geq i+1} p_k u(x_k)\\ &< \sum_{k\leq i} p_k(x_k)(-u(-x_k)) + \sum_{k\geq i+1} p_k u(x_k)\\ &= -\sum_{k\geq i+1} p_k u(x_k) + \sum_{k\geq i+1} p_k u(x_k) = 0\end{aligned}$$

($\Rightarrow$): Suppose that $\exists x^* > 0 : u(x^*) \geq -u(-x^*)$. Define $l = (-x^*, \frac{1}{2}; x^*, \frac{1}{2})$. Then we have:

$$\begin{aligned}\mathbb{E}u(l) &= \tfrac{1}{2}u(-x^*) + \tfrac{1}{2}u(x^*)\\ &\geq -\tfrac{1}{2}u(x^*) + \tfrac{1}{2}u(x^*) = 0\end{aligned}$$

A contradiction □

Observe that for prospects $l$ such that $l = -l$, $\mathbb{E}l = 0$. By the fact that $\mathbb{E}u(l) < 0$, $CE(l) < \mathbb{E}l = 0$. This shows that loss aversion is a special kind of risk aversion. It might be a useful hint in decomposing risk attitudes and loss attitudes.

# 5 Discussion

# 6 Dutch book arguments

Dutch books were analyzed by Yaari (1985).

Dual theory of choice under risk was introduced by Yaari (1987). The crucial axiom of this theory is dual independence. Let $l, l', l'' \in \Delta F$ be three prospects and $\alpha \in [0, 1]$. Let $p_{l,l'} : (\Delta X)^2 \to [0, 1]$ be the joint distribution of $l$ and $l'$. Define prospect $\alpha l + (1-\alpha)l' \in \Delta F$ such that $p_{\alpha l+(1-\alpha)l'}(\alpha x + (1-\alpha)y) = p_{l,l'}(x, y)$, where $(x, y) \in (\Delta X)^2$. In other words $\alpha l + (1-\alpha)l'$ denotes the usual sum of two random variables $\alpha l$ and $(1-\alpha)l'$. It is different than in the statement of vNM independence where the corresponding object is a probability mixture. Let $\succsim \in (\Delta F)^2$ be a preference relation for prospects. It satisfies dual independence if $l \succsim l'$ implies $\alpha l + (1-\alpha)l'' \succsim \alpha l' + (1-\alpha)l''$.

A Dutch book is a sequence of trades that when accepted leads to a sure loss of money for the accepting party and a sure win for the proposing party. It is a well established fact that if the decision maker violates any of the Expected Utility axioms or any of the probability axioms, he is vulnerable to a Dutch book (see Yaari, 1985) Dutch Books numbered DB1, DB2 and DB3. Yaari suggested that if the decision maker violates dual independence but does not violate any of the Expected Utility and probability axioms, he is also vulnerable to a special kind of Dutch book that he calls DB4. Let us analyze his argument in greater detail. Suppose that the decision maker exhibits the following pattern of preferences:

$$l \succ l' \text{ and } \tfrac{1}{2}l' \succ \tfrac{1}{2}l. \tag{14}$$

Then a bookie may approach him and offer him the following deal:

1. Take $l/2$ and $l'/2$ free of charge.
2. Exchange $l/2$ for $l'/2$ and pay me a small positive amount $\epsilon_1$.
3. Note that you now own 2 halves of $l'$, which is the same as one $l'$. Exchange it for $l$ and pay me a small positive amount $\epsilon_2$.
4. Note that one $l$ is the same as two halves of $l$. Exchange $l/2$ for $l'/2$ and pay me a small positive amount $\epsilon_3$.
5. Note that you are back to where you were Step 1. except for having lost $\epsilon_1 + \epsilon_2 + \epsilon_3$ along the way. Repeat steps 2.-4.

Usually one thinks of a sequence of prospects in the following way. At one time period, a single prospect is offered and either accepted or rejected. Subsequently, the prospect is realized, and in the next step another prospect is offered. However, the way the sequence of trades above is constructed makes it clear that we have to consider each successive step without the corresponding prospects being realized. What is happening here is trading with prospects themselves. Therefore we needed a framework in which we consider accepting a given prospect while already possessing the right to play another prospect. The situation is thus more complicated than usual because we have to consider background risk, i.e. our initial position is allowed to be random.

We will now analyze Yaari argument in greater detail. We assume that before any exchange has taken place, all the uncertainty is resolved so that the decision maker's initial wealth is certain $W$.

TODO: This should be analyzed together with point a) below

Let us first consider the Expected Utility of wealth model. For the Dutch book to work we would need the following pattern of preferences:

$$W + l/2 + l'/2 \succ W + l \succ W + l' \succ W + l/2 + l'/2$$

And this is impossible as it implies a cycle excluded by transitivity.

Let us now consider the Expected Utility of wealth changes model. In this model it is crucial first to specify a way to determine reference act $h$ in any choice situation. We will consider several possibilities.

a) Reference act $h_1$ is each time the certain initial wealth $W$.

b) Reference act $h_2$ is a common part to both alternative acts $f$ and $g$

c) Reference act $h_3$ is the current status quo wealth

Let us now examine Yaari's Dutch book no. 4 under the above three scenarios. Let $h_1, h_2, h_3$ be three alternative reference acts as specified in a)-c) above and in each step let $f$ be a status quo act and $g$ an alternative act for which $f$ is exchanged.

| Steps | $h_1$ | $h_2$ | $h_3$ | $f$ | $g$ |
|---|---|---|---|---|---|
| 1. | $W$ | $W$ | $W$ | $W$ | $W + l/2 + l'/2$ |
| 2. | $W$ | $W + l'/2$ | $W + l/2 + l'/2$ | $W + l/2 + l'/2$ | $W + l'$ |
| 3. | $W$ | $W$ | $W + l'$ | $W + l'$ | $W + l$ |
| 4. | $W$ | $W + l/2$ | $W + l$ | $W + l$ | $W + l/2 + l'/2$ |

Under the specified assumptions, the following conditions should hold in each of the three scenarios in order for the Dutch book to work:

| Steps | $h_1$ | $h_2$ | $h_3$ |
|---|---|---|---|
| 1. | $\mathbb{E}u(l/2 + l'/2) > 0$ | $\mathbb{E}u(l/2 + l'/2) > 0$ | $\mathbb{E}u(l/2 + l'/2) > 0$ |
| 2. | $\mathbb{E}u(l') > \mathbb{E}(l/2 + l'/2)$ | $\mathbb{E}u(l'/2) > \mathbb{E}(l/2)$ | $\mathbb{E}u(l'/2 - l/2) > 0$ |
| 3. | $\mathbb{E}u(l) > \mathbb{E}u(l')$ | $\mathbb{E}u(l) > \mathbb{E}(l')$ | $\mathbb{E}u(l - l') > 0$ |
| 4. | $\mathbb{E}u(l/2 + l'/2) > \mathbb{E}u(l)$ | $\mathbb{E}u(l'/2) > \mathbb{E}(l/2)$ | $\mathbb{E}u(l'/2 - l/2) > 0$ |

It is clear that the case a) does not allow DB no. 4 as it is in fact equivalent to the Expected Utility of wealth model with zero initial wealth.

Let us focus on cases b) and c).

**Proposition 6.1.** *If the reference act is chosen to be a common part to the alternatives $f$ and $g$, then Dutch book no. 4 is possible if and only if the utility function $u$ does not belong to the CRRA class.*

*Proof.* That DB no. 4 is impossible under CRRA follows directly from a well-known [1] characterization result on the CRRA class: The utility function ex-

[1] See Pratt (1964) as the original reference or Lewandowski (2013) for a consise treatment.

hibits CRRA if and only if the following holds:

$$l \succ l' \iff \lambda l \succ \lambda l', \quad \forall \lambda > 0, \; \forall l, l' \in \Delta F$$

It is clear that under CRRA one cannot obtain (14).

If the utility function is not CRRA, then take any two nondegenerate lotteries $l, l' \in \Delta F$ such that $l \sim l'$. Then by the above characterization result for CRRA, there must be $\lambda > 0$ such that $\lambda l \nsim \lambda l'$. W.l.o.g. we assume that $\lambda l \prec \lambda l'$.

a) If it is true for $\lambda = \frac{1}{2}$, we find $\epsilon > 0$ small enough such that shifting probability mass from some element in the support of $l$ upwards and redefine the resulting new lottery as $l$ will result in $l \succ l'$ and $\lambda l \prec \lambda l'$. It is always possible to find such $\epsilon$ by Continuity axiom of Expected Utility and monotonicity wrt First Order Stochastic Dominance.

b) If it is true for $\lambda \neq \frac{1}{2}$, then ...

□

We will now give a simple example of a utility function which is not CRRA and thus DB no. 4 works. Take a CARA utility function of the form: $u(x) = 10 - 10 \exp^{-\frac{1}{10}x}$, where $x \in \Delta X$ and two prospects

$$l := (8, 0.9; 0, 0.1), \; l' := (20, 0.5; 0, 0.5).$$

Then the following holds: $\mathbb{E}u(l) > \mathbb{E}u(l')$ and $\mathbb{E}u\left(\frac{1}{2}l'\right) > \mathbb{E}u\left(\frac{1}{2}l\right)$, which represents preference pattern given by (14).

Let us now analyze case c).

**Proposition 6.2.** *If the reference act is chosen to be a current status quo act, then Dutch book no. 4 is possible if and only if the utility function $u$ is not concave.*

*Proof.* To prove the proposition we will use the following result:

**Lemma 4.** *For a concave utility function $\mathbf{x} \succ \mathbf{0} \Rightarrow -\lambda\mathbf{x} \prec \mathbf{0}$, for $\lambda > 0$ and any $\mathbf{x}$*

First, let's state two properties valid for any concave function $u(\cdot)$ such that $u(0) = 0$:

- for $\lambda \in (0, 1]$ the following holds $\lambda u(x) \leq u(\lambda x)$, for all $x$.

- for any $x$, $u(x) \leq -u(-x)$.

The first property is a version of Jensen's inequality whereas the second may be easily verified. Using these two properties, we establish the following relationship, true for all concave functions:

$$\lambda u(x) \leq u(\lambda x) \leq -u(-\lambda x) \leq -\lambda u(-x), \quad \forall \lambda \in (0,1], \forall x \tag{15}$$

Now suppose we take any $x$ and $\lambda \in (0,1]$ such that the above statement is true. Define $y = \lambda x$ and $\theta = \frac{1}{\lambda}$ and divide the above sequence of inequalities by $\lambda$. Notice that $x$ and $\lambda$ may be chosen so that $y$ is anything. Notice further that $\theta \in [1,+\infty)$ since $\lambda \in [0,1)$. Hence we obtain the following relationship valid for any concave function:

$$u(\theta y) \leq \theta u(y) \leq -\theta u(-y) \leq -u(-\theta y), \quad \forall \theta \in [1,+\infty), \forall y \tag{16}$$

Having established the two relationships, we can proceed with the proof. Suppose that $\mathbf{x} \succ \mathbf{0}$. It follows that $\mathrm{E}u(\mathbf{x}) > 0$. And by (15) it is true that:

$$\mathrm{E}u(-\lambda\mathbf{x}) \leq -\lambda \mathrm{E}u(\mathbf{x}) < 0, \quad \forall \lambda \in (0,1], \forall \mathbf{x}$$

It follows therefore that $-\lambda\mathbf{x} \prec \mathbf{0}$, for $\lambda \in (0,1]$. By (16) it is true that:

$$-\mathrm{E}u(-\lambda\mathbf{x}) \geq \lambda \mathrm{E}u(\mathbf{x}) > 0, \quad \forall \lambda \in [1,+\infty), \forall \mathbf{x}$$

It follows that $\mathrm{E}u(-\lambda\mathbf{x}) < 0$ and hence $-\lambda\mathbf{x} \prec \mathbf{0}$ for $\lambda \in [,+\infty)$.
We have shown that $\mathbf{x} \succ \mathbf{0} \Rightarrow -\lambda\mathbf{x} \prec \mathbf{0}$ for any $\mathbf{x}$ and for $\lambda$ both in $(0,1]$ and in $[1,+\infty)$. Hence it is true for all positive $\lambda$. □

The above proposition makes it clear that if the utility function is concave, it is not possible to have both $\mathbf{x} \succ \mathbf{0}$ and $-\lambda\mathbf{x} \succ \mathbf{0}$ for $\lambda > 0$. It means that the risk averse individual will never accept both short and long position on the same gamble or a share thereof. For a convex utility function on the other hand such a situation is possible. Consider for example a strictly convex utility function and a gamble $\mathbf{x} = (-x, 1/2; x, 1/2)$. Since the utility function is strictly convex we have that $\mathbf{x} \succ \mathbf{0}$. Observe that $\mathbf{x} = -\mathbf{x}$ so that $-\mathbf{x} \succ \mathbf{0}$. The same will be true of any multiple of $-\mathbf{x}$, i.e. $-\lambda\mathbf{x} \succ \mathbf{0}$, for all $\lambda > 0$.

# 7 TGPT model

Now we present the $\mathrm{PT}^3$ model which departs not only from the consequentialist's assumption but also from the EU hypothesis. Let the states be denoted by $s_i$ and ordered according to the value of $v(f(s_i), h(s_i))$. Out of all the states ($|S| \geq 2$) let $m^+ \geq 0$ be the number of states for which $v(f(s_i), h(s_i)) \geq 0$ (relative weak gain) and $m^- \geq 0$ be the number of states for which $v(f(s_i), h(s_i)) < 0$ ( relative loss). Let the states with relative loss be indexed by $-m^-, ..., -1$ and let the states with relative gain be indexed by $1, ..., m^+$. Let the probabilities $\pi(s_i)$ be denoted by $\pi_i$.

The PT[3] introduces rank-dependent probability weighting and thus the function $V(f, h)$ representing the reference-dependent preference becomes:

$$V(f, h) = \sum_i v(f(s_i), h(s_i))w(s_i; f, h) \tag{17}$$

where $w(s_i; f, g)$ is the (rank-dependent) decision weight assigned to $s_i$ when $f$ is being viewed from $h$. With the indexing proposed above this weight is calculated according to:

$$w(s_i; f, h) = \begin{cases} w^+ \left(\sum_{j \geq i} \pi_j\right) - w^+ \left(\sum_{j > i} \pi_j\right), & 1 \leq i \leq m^+ \\ w^- \left(\sum_{j \leq i} \pi_j\right) - w^- \left(\sum_{j < i} \pi_j\right), & -m^- \leq i \leq -1 \end{cases} \tag{18}$$

with the convention that $w^+ \left(\sum_{j > m^+} \pi_j\right) = w^- \left(\sum_{j < m^-} \pi_j\right) = 0$.

The PT[3] model encompasses other models as special cases. Cumulative prospect Theory is obtained if $v(f(s_i), h(s_i)) = u(f(s_i) - h(s_i))$, where $u$ is a strictly increasing and continuous value function for which $u(0) = 0$ and the reference act is a degenerate act $h = f^x$. If $w(s_i; f, g) = \pi_i$, then it reduces back to the RDSEU model. If the reference act is the current wealth position then we have the expected utility of income (or wealth changes) model (See Sadiraj, Cox, 2006, or Palacios-Huerta, Serrano, 2006, Lewandowski, 2014).

Note that in the models given by (1) and (17), states of nature $s$ are needed only in order to define a joint distribution of $f$ and $h$ so that we can define values of the function $v$. We can thus simplify the models as soon as we know the joint distribution. For two acts $f, h$ let $p_{f,h} : X^2 \to [0, 1]$ be the induced joint probability distribution of $f, h$, such that $p_{f,g}(x, y) = \sum_{\{s \in S: f(s)=x, h(s)=y\}} \pi(s)$, for all $x, y \in X^2$. Given that we can simplify the model given by (1):

$$V(f, h) = \sum_{(x,y) \in X^2} v(x, y)p_{f,h}(x, y) \tag{19}$$

Let us denote by $x_i, y_i$ the corresponding values of $f$ and $h$ and by $\pi_i$ the probabilities $p_{f,h}(x_i, y_i)$. Order these triples increasingly according to the value of $v(x_i, y_i)$ and divide the indexes into the set $1, ..., m^+$ for which the value

$x_i - y_i$ is nonnegative and the set $-m^-, ..., -1$ for which the value $x_i - y_i$ is negative. Then we can simplify the model given by (17) into:

$$V(f,h) = \sum_i v(x_i, y_i) w(\pi_i; f, h) \tag{20}$$

where $w(\pi_i; f, h)$ is a probability weighting function defined in the same way as (18) but with indexes $i$ referring to the ordering of $\pi_i$ and $x_i, y_i$ instead of the ordering of states $s_i$ given there.

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