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Complementary symmetry in Cumulative Prospect Theory with random reference

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HIGHLIGHTS

- We study Cumulative Prospect Theory models of buying and selling prices.
- In Model 1 the gramble's prizes are integrated with the price.
- In Model 1 complementary symmetry holds for any kind of loss or risk attitude.
- In Model 2 the utility of a gamble is balanced against the price.
- Constant buying/selling price ratio in Model 2 relies on preference homogeneity.

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ABSTRACT

It is shown that complementary symmetry holds in Cumulative Prospect Theory with random reference if the utility function for gains and losses is strictly increasing and continuous. Previous results imposed more restrictions involving preference homogeneity, reflection, and loss aversion. The result holds true in the general version of the Third-Generation Prospect Theory provided that the relative value function v takes the same form as in Cumulative Prospect Theory.

1. Introduction

Complementary symmetry is a property introduced by Birnbaum and Zimmermann (1998). It involves two binary gambles $g := (x, p; y)$ and $g' := (x, 1 - p; y)$, where x, y are two monetary outcomes such that $x > y$, and $p \in (0, 1)$ is the probability of receiving x in g and y in g'. It says that the sum of the buying price of g – i.e. the largest amount an individual is willing to pay for g, denoted by $b(g)$ – and the selling price of g' – i.e. the smallest amount an individual is willing to accept to forfeit g', denoted by $s(g')$ – equals the sum of the two monetary outcomes: $x + y$.

This property has been shown to fail in experimental settings (Birnbaum and Sutton, 1992; Birnbaum, Yeary, Luce, and Zhao, 2016, and Birnbaum and Zimmermann, 1998). The experiments were designed to elicit buying and selling prices for each individual in a group of subjects for a series of binary gambles g, g' where the amount $x + y$ was held constant. It was found that the sum of the median $b(g)$ value and the median $s(g')$ value is not constant and depends on the range $x - y$. The sum is always below the value $x + y$ and decreases as the range becomes wider. For example, Birnbaum and Sutton (1992) show that the median buying and selling prices of (\$60, 0.5; \$48) are \$50 and \$54, respectively, and thus their sum equals \$104. However, the median buying and selling prices of (\$96, 0.5; \$12) are \$25 and \$50, respectively, and their sum equals a meager \$75.

In the buying/selling price elicitation task the decision maker is asked to make a trade-off between the gamble in question and a sure amount of money that the gamble is exchanged for. The seller exchanges the gamble he owns for a sure amount of money whereas the buyer exchanges the money in his possession for the gamble he wants to acquire. In order to model this kind of asymmetric trade-off within Cumulative Prospect Theory (CPT in short, Tversky and Kahneman, 1992) Birnbaum and Zimmermann (1998) (Appendix B) proposed two models. In model 1 the utility of a gamble is balanced against the price (obtained when selling

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or paid when buying). This model is an extension of the model of Tversky and Kahneman (1991) that was proposed for goods to gambles. In Model 2 the gamble's monetary prizes are integrated with the price: the price serves as reference to evaluate the gamble when buying and the gamble serves as (random) reference to evaluate the price when selling. Birnbaum and Zimmermann (1998) identified the key implications of each of the two models and have shown that they are inconsistent with the experimental evidence suggesting that the range of outcomes, i.e. $|x-y|$, plays an important role in the price elicitation tasks (see for instance Birnbaum and Beeghley, 1997; Birnbaum and Stegner, 1979; Birnbaum and Sutton, 1992). In the case of Model 2 the questionable implication identified by Birnbaum and Zimmermann (1998) is complementary symmetry, whereas in the case of Model 1 it is constant selling to buying price ratio.

The purpose of this note is to show whether these implications carry over to the case where we relax some of the strong assumptions of the parametric CPT model that were used in Birnbaum and Zimmermann (1998). We shall mainly focus on Model 2 (and hence on complementary symmetry) because the main idea of this model, i.e. the integration of prizes and prices, has become standard in later accounts (e.g. Luce, 1991) and, in particular, has been adopted in the Third-Generation Prospect Theory (PT^3 in short, Schmidt, Starmer, and Sugden, 2008). We study the less popular Model 1 and its implication of constant selling to buying price ratio in Appendix.

Within model 2 Birnbaum et al. (2016) and Birnbaum and Zimmermann (1998) were able to demonstrate that, irrespective of the form of the probability weighting functions for gains and losses, complementary symmetry holds if utility function for gains and losses is of the following form:

$$u(x)=\begin{cases} x^{\alpha}, & \text{for } x\geq 0,\\ -\lambda(-x)^{\alpha}, & \text{for } x<0. \end{cases} \text{ where } \alpha>0. \tag{1}$$

The main contribution of this note is to show that complementary symmetry holds more generally in this model for any strictly increasing and continuous utility function satisfying $u(0)=0$. In particular, it holds irrespective of whether any kind of loss aversion, or reflection,[1] or preference homogeneity (power utility) is assumed. Section 2 introduces the model and proves the main result. Section 3 shows how the results are carried over to the more general PT^3 model. Appendix analyses the implications of Model 1 of Birnbaum and Zimmermann (1998).

2. Complementary symmetry in cumulative prospect theory with random reference

Except for a few exceptions, we adopt the same setup and notation as in Birnbaum et al. (2016) to enhance comparability. Let $(x,p;y)$ be a binary prospect, in which the outcome is x with probability $p\in(0,1)$ and y otherwise, where $x,y\in\mathbf{R}$ and $x>y$. It is assumed that outcomes x and y are defined relative to some reference outcome that is normalized to zero; a negative outcome is thus perceived as a loss and a positive one as a gain. The CPT model for $(x,p;y)$ is then written as follows:

$$U(x,p;y) = \begin{cases} u(x)w^{+}(p)+u(y)[1-w^{+}(p)], & \text{if } x>y\geq 0,\\ u(x)w^{+}(p)+u(y)w^{-}(1-p), & \text{if } x\geq 0\geq y,\\ u(x)[1-w^{-}(1-p)]+u(y)w^{-}(1-p), & \text{if } 0\geq x>y \end{cases} \tag{2}$$

[1] It can be proved for CPT that changing the sign of all consequences of two prospects always reverses the preference is equivalent to assuming the (strictly increasing and continuous) utility function of the following form: $u(0)=0$, $u(x)=-\lambda u(-x)$, for all $x>0$, where $\lambda>0$.

where $u:\mathbf{R}\to\mathbf{R}$ is a strictly increasing and continuous utility (value) function satisfying $u(0)=0$, and $w^{+}:[0,1]\to[0,1]$, $w^{-}:[0,1]\to[0,1]$ are strictly increasing and continuous probability weighting functions for gains and losses, respectively, satisfying $w^{+}(0)=w^{-}(0)=0$ and $w^{+}(1)=w^{-}(1)=1$.

A gamble is the same as a prospect except that a former may not be defined relative to a reference outcome whereas the latter always is. In the CPT model we subtract a reference outcome from each outcome of a gamble to form a prospect. In what follows we allow the possibility of a random reference outcome. Generally, this would require the state-space approach such as in the PT^3 model in order to take into account the dependence structure between an evaluated gamble and a reference gamble. However, when modeling the buying and the selling prices, it is never the case that a reference and an evaluated object are both random.[2] Hence, to keep things simple, we will stick to the simple setup of gambles being probability distributions. In the non-standard case of evaluating a sure outcome relative to a random reference, we shall form a prospect by subtracting each outcome of the gamble from the sure alternative in each state.

Consider a gamble $g:=(x,p;y)$, where $x>y$. We define the buying price $b(g)\in\mathbf{R}$ as the maximal sure outcome for which the decision maker is willing to buy g when it is evaluated relative to the reference outcome $b(g)$. Similarly we define the selling price $s(g)\in\mathbf{R}$ as a minimal sure outcome for which the decision maker is willing to sell g when $s(g)$ is evaluated relative to g. The prices thus satisfy the following conditions:

$$U[x-b(g),p;y-b(g)]=0, \tag{3}$$
$$U[s(g)-x,p;s(g)-y]=0. \tag{4}$$

Having defined the model we now state the main result.

Proposition 2.1. *Consider two gambles $g:=(x,p;y)$ and $g':=(x,1-p;y)$, where $x>y$, and $p\in(0,1)$. In the model defined by (2), (3) and (4) the following holds:*

$$b(g)+s(g')=x+y. \tag{5}$$

Proof. To save on notation denote $s:=s(g')$ and $b:=b(g)$. Note first that by (2), monotonicity of u and the fact that $u(0)=0$, it must be that both prices lie between the lower and the upper outcome of the corresponding prospect, i.e. $b,s\in(y,x)$. Hence they satisfy:

$$w^{+}(p)u(x-b)+w^{-}(1-p)u(y-b)=0$$
$$w^{-}(1-p)u(s-x)+w^{+}(p)u(s-y)=0.$$

Or after rearranging and combining:

$$\frac{u(s-y)}{-u(s-x)}=\frac{u(x-b)}{-u(y-b)}=\theta(p), \tag{6}$$

where $\theta(p):=\frac{w^{-}(1-p)}{w^{+}(p)}$. Suppose now that contrary to the claim it is not true that $b+s=x+y$. There are two cases to consider:

[2] One can argue, however, that the buying or the selling price is not a point-estimate but either a random variable or a fuzzy number. The intuition behind is that it is often difficult to choose a crisp numerical value below which the decision maker will not sell (or above which she will not buy). It may be that there is an interval of prices $[s_l,s_u]$, $s_l<s_u$ such that the decision maker is sure that she would not sell below s_l and is sure she would sell above s_u, and she remains hesitant in between s_l and s_u. While we believe that this is a valid possibility, we decided not to follow this path, as it would require a different preference structure, allowing for instance the violations of completeness or of transitivity of indifference.

1. Suppose that $b + s < x + y$. Then we get the following sequence of equivalences:

$$\begin{aligned} & x - b > s - y \\ \iff\quad & u(x-b) > u(s-y) && \text{By monotonicity of } u \\ \iff\quad & \frac{-u(s-y)u(y-b)}{-u(s-x)} > u(s-y) && \text{By (6)} \\ \iff\quad & -u(y-b) > -u(s-x) && \text{Since } u(s-y) > 0 > u(s-x) \\ \iff\quad & u(y-b) < u(s-x) \\ \iff\quad & y - b < s - x && \text{By monotonicity of } u \end{aligned}$$

A contradiction.

2. Suppose that $b + s > x + y$. A similar argument leads to a contradiction as well.

Hence it must be that $b + s = x + y$, as claimed. □

3. Complementary symmetry in the third-generation prospect theory model

How does the above result carry over to the PT^3 model of Schmidt et al. (2008)? The PT^3 model is based on the Reference-Dependent Subjective Expected Utility model (RDSEU in short, Sugden, 2003). It requires the finite state space S and the Savage acts $f, g, h \in F$ (functions from S into the outcome space X). The general version of PT^3 represents a reference-dependent preference relation via the function V: $f \succsim_h g$ (f is preferred to g when viewed from a reference act h) if and only if $V(f, h) \geq V(g, h)$, where $V(f, h)$ (and similarly $V(g, h)$) are defined as:

$$V(f,h) = \sum\nolimits_{s \in S} v(f(s), h(s)) W(s; f, h)$$

where $W(\cdot; f, h) : S \to [0, 1]$ is the decision weight assigned to state s when f is viewed from h; $v : X^2 \to \mathbf{R}$ is a relative value function that is strictly increasing in its first argument, with $v(x, y) = 0$ when $x = y$.

The parametric form of PT^3 restricts the general form using several assumptions. The first two are the assumptions that also hold in CPT[3]:

1. The relative value function v takes the form: $v(x, y) = u(x - y)$, where u is a strictly increasing and continuous utility function satisfying $u(0) = 0$.
2. The decision weights $W(s; f, h)$ are the same as in CPT, i.e. are derived from cumulative probability weighting functions[4] for gains $w^+ : [0, 1] \to [0, 1]$ and for losses $w^- : [0, 1] \to [0, 1]$.

Then, there are two more specific parametric assumptions that restrict the shape of the functions u and w^+, w^-:

1.a The utility function u is of the form specified in (1).
2.a The probability weighting functions w^+ and w^- are the same and take the following form: $w(\pi) = \frac{\pi^\beta}{(\pi^\beta + (1-\pi)^\beta)^{1/\beta}}$, with $\beta > 0$, where π is a (cumulative) probability.

In which of the described models does complementary symmetry hold? First, it holds irrespective of the form of decision weights. Hence it requires neither the structural assumption 2. nor the parametric assumption 2.a. It means that complementary symmetry holds both if the decision weights are as in CPT and if they are simply probabilities as in the RDSEU model. Second, the difference between the model of Birnbaum et al. (2016) and Birnbaum and Zimmermann (1998) and the model analyzed in this note is that the former requires both the structural assumption 1. as well as the parametric assumption 1.a. whereas the latter requires only assumption 1. Summing up, complementary symmetry holds in the general PT^3 model with the additional assumption 1 imposed on the relative value function to make it consistent with CPT.

Since complementary symmetry has persistently failed in experimental settings, the result of this note casts doubts on the relevance of the CPT model with random reference (as well as the most widely used version of the PT^3 model) as a good descriptive model of buying and selling prices of risky gambles.

Appendix. The implications of the alternative model balancing prices against utilities

In this section we will analyze Model 1 from Appendix B of Birnbaum and Zimmermann (1998). We adopt the same notation and assumptions as in Section 2. We show here that unlike complementary symmetry, constant buying to selling price ratio does not carry over to the case where we relax some of the strong assumptions of the parametric CPT model used in Birnbaum and Zimmermann (1998).

Given the gamble $g := (x, p; y)$ we define the buying price $b(g) \in \mathbf{R}$ (similarly, the selling price $s(g) \in \mathbf{R}$) by balancing off the utility of gaining (losing) a gamble and the utility of paying (being paid) the price. It may be written in the following way:

$$-U[-b(g)] = U[x, p; y], \tag{7}$$

$$U[s(g)] = -U[-x, p; -y]. \tag{8}$$

For simplicity we assume that g is a gain gamble (i.e. $x, y \geq 0$), but an analogous analysis can easily be done for a loss gamble or a mixed gamble as well.[5]

Note that the gamble is evaluated separately from the prices and the reference point for evaluating both the gamble and separately each of the prices is always 0—the sure *status quo* outcome. In the model given by (7), (8) and (2), Birnbaum et al. (2016) and Birnbaum and Zimmermann (1998) have shown that if the utility function u takes the form (1) and that weighting functions for gains and losses agree, (i.e. $w^+ = w^-$), then $s(g)$ is a constant multiple of $b(g)$. Furthermore, $s(g)$ is greater than $b(g)$ if and only if $\lambda > 1$ (which is equivalent to loss aversion). However, as reported by Birnbaum and Zimmermann (1998) there is strong evidence that the ratio of selling to buying price, while consistently greater than one, is not constant, but rather varies with the range of lottery outcomes $|x - y|$. We shall now demonstrate that by relaxing the assumption (1) we may obtain ratios of buying to selling prices that may vary with $|x - y|$.

We first note that the utility function given by (1) is equivalent in the present setup to assuming preference homogeneity, i.e. if each consequence of a prospect (either mixed of not) is multiplied by a positive constant, then so is the cash equivalent of this prospect (see Proposition 4.3 in Lewandowski, 2017). In what follows we will consider weaker assumptions. We shall start by

[3] They should not be called parametric, as they do not impose any specific parameters; instead they impose functional restrictions of a general kind.

[4] The form of rank-dependent decision weights used in CPT is standard and hence is not invoked here.

[5] By assuming a gain gamble we make sure that the buying and selling prices are both non-negative. Otherwise, we would have to consider the possibility of paying a negative price that is equivalent to being paid a positive price. We want to avoid this unnecessary complication that does not add much to the argument presented here.

replacing preference homogeneity with the following very general form of loss aversion[6]:

Definition 1. Loss aversion holds if the *status quo* outcome (i.e. outcome 0) is strictly preferred to the binary prospect $(x, 0.5; -x)$, for any $x \neq 0$.

Proposition A.1. *Assume the model given by (2) with $w^+ = w^- =: w$, and (7), (8). Then loss aversion holds if and only if $s(g) > b(g)$ for any non-degenerate (single-element support gambles are excluded) binary gamble g.[7] Moreover, the ratio of selling to buying price may vary with the range of outcomes of the binary gambles.*

Proof. It is straightforward to verify that given the CPT model with $w^+ = w^- =: w$ loss aversion is equivalent to the following condition: $u(x) < -u(-x)$ for $x \neq 0$. We first prove the $\Rightarrow$ direction of the first part of the proposition. Consider any non-degenerate binary gamble g. Suppose $b(g) \neq 0$ and $b(g) \geq s(g)$ contrary to the claim. Then by loss aversion and monotonicity of u we have: $-u(-b(g)) > u(b(g)) \geq u(s(g))$.

On the other hand, writing the definitions (7) and (8) for the CPT model (2) we get:

$$-u(-b(g)) = w(p)u(x) + (1 - w(p))u(y) \tag{9}$$

$$u(s(g)) = -w(p)u(-x) - (1 - w(p))u(-y). \tag{10}$$

By loss aversion the RHS of (9) is smaller than the RHS of (10). Thus the LHS of (9) must be smaller than the LHS of (10), i.e.: $-u(-b(g)) < u(s(g))$. Thus we have a contradiction. It is left to check the case $b(g) = 0$. Then by (9), (10), loss aversion, monotonicity of u and the fact that the gamble is non-degenerate we have that $s(g) > 0 = b(g)$.

Now we prove the $\Leftarrow$ direction of the first part of the proposition. The contraposition states that if $u(x) \geq -u(-x)$ for any x (loss seeking) then $b(g) \geq s(g)$. Suppose $b(g) < s(g)$ contrary to the claim. Then by monotonicity of u: $u(-b(g)) > u(-s(g))$.

On the other hand, by loss seeking the RHS of (9) is not smaller than the RHS of (10). It means that the LHS of (9) is also not smaller than the LHS of (10), i.e. $-u(-b(g)) \geq u(s(g))$. Using loss seeking we get $u(s(g)) \geq -u(-s(g))$. Hence we get $u(-b(g)) \leq u(-s(g))$, a contradiction. This finishes the proof of the first part of the proposition.

Now we prove the second part of the proposition. The ratio of selling to buying price can be written as:

$$\frac{s(g)}{b(g)} = \frac{u^{-1}\left(-w(p)u(-x) - (1 - w(p))u(-y)\right)}{-u^{-1}\left(-w(p)u(x) - (1 - w(p))u(y)\right)}. \tag{11}$$

It is readily verified that the ratio may vary with x and y and with $|x - y|$. □

Note that if we assume preference homogeneity as in (1) then the ratio becomes:

$$\frac{s(g)}{b(g)} = \gamma^{\frac{2}{\alpha}}.$$

And hence is constant as shown by Birnbaum and Zimmermann (1998). On the other hand, if we replace preference homogeneity with strong reflection (it is equivalent to assuming the following continuous and strictly increasing utility function: $u(0) = 0$, $u(x) = -\lambda u(-x)$, for all $x > 0$, where $\lambda > 0$)[8] then the ratio becomes:

$$\frac{\bar{u}(s(g))}{\bar{u}(b(g))} = \lambda^2.$$

What is constant here is not the ratio of selling to buying price but the ratio of utilities of the two prices. This formulation also allows the ratio of prices to vary with $|x - y|$ but in a more restricted way than in (11). Finally, we may consider the utility function of the following form:

$$u(x) = \begin{cases} x^\alpha, & \text{for } x \geq 0, \\ -\lambda(-x)^\beta, & \text{for } x < 0. \end{cases} \quad \text{where } \alpha, \beta > 0. \tag{12}$$

The above utility function differs from (1), i.e. the preference homogeneity case, in that it has different curvature parameters for losses and for gains (β and α, respectively). This is in fact the form postulated by CPT,[9] and not (1). In the general case, i.e. when $\alpha \neq \beta$, the utility function (12) neither satisfies preference homogeneity (it satisfies it separately for gain prospects and separately for loss prospects but not for mixed prospects[10]) nor strong reflection. However, if $\lambda > 1$ it exhibits loss aversion defined by Definition 1. The ratio of selling to buying price for this utility function becomes:

$$\frac{s(g)}{b(g)} = \frac{\left(w(p)x^\beta + (1 - w(p))y^\beta\right)^{\frac{1}{\alpha}}}{\left(w(p)x^\alpha + (1 - w(p))y^\alpha\right)^{\frac{1}{\beta}}} \lambda^{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

This ratio is also not constant and may vary with the outcomes of the gamble.

Summing up, unlike complementary symmetry in Model 2, the ratio of selling to buying price is not constant if we relax the assumptions adopted by Birnbaum and Zimmermann (1998). In particular, even if we assume the utility function (12) postulated in the parametric CPT model the ratio may vary with the outcomes as well as the range of outcomes of the gamble.

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6 There are other more restrictive definitions, see e.g. Lewandowski (2017).

7 The result easily generalizes to multi-outcome gambles.

8 See proposition 4.5 in Lewandowski (2017).

9 Tversky and Kahneman (1992), equation (5).

10 Contrary to what is suggested in Tversky and Kahneman (1992), Section 2.3, page 309.