Range-Dependent Utility

Michał Lewandowski
joint work with Krzysztof Kontek

March 23, 2017

Outline

1. Inspired by Parducci (1964) we propose **range-dependent utility (RDU)** as a general framework for decisions under risk
   - Simple modification of Expected Utility Theory in which utility depends on the range of lottery outcomes
2. Based on TK 1992 experimental data we propose the **decision utility model (DU)** as operational special case of RDU used for prediction:
   - The model is based on the hypothesis that preferences are scale and shift invariant
3. Monotonicity wrt FOSD and continuity
   - Necessary and sufficient conditions
   - Examples

Eye-adaptation process

What we see in a dark room

Just after entering After 15 minutes

What an eye adapts to:

- Mean luminance level
- Or luminance range?

Two psychophysical theories:

- Adaptation-level theory (Helson 1963) $\Rightarrow$ **reference point** $\Rightarrow$ Prospect Theory
- Range-frequency theory (Parducci 1964) $\Rightarrow$ **range** $\Rightarrow$ Our model

TK (1992) data

No	xl	xu	p	CE
1	0	50	0.10	9.0
2	0	50	0.50	21.0
3	0	50	0.90	37.0
4	0	100	0.05	14.0
5	0	100	0.25	25.0
6	0	100	0.50	36.0
7	0	100	0.75	52.0
8	0	100	0.95	78.0
9	0	200	0.01	10.0
10	0	200	0.10	20.0
11	0	200	0.50	76.0
12	0	200	0.90	131.0
13	0	200	0.99	188.0
14	0	400	0.01	12.0
15	0	400	0.99	377.0
16	50	100	0.10	59.0
17	50	100	0.50	71.0
18	50	100	0.90	83.0
19	50	150	0.05	64.0
20	50	150	0.25	72.5
21	50	150	0.50	86.0
22	50	150	0.75	102.0
23	50	150	0.95	128.0
24	100	200	0.05	118.0
25	100	200	0.25	130.0
26	100	200	0.50	141.0
27	100	200	0.75	162.0
28	100	200	0.95	178.0

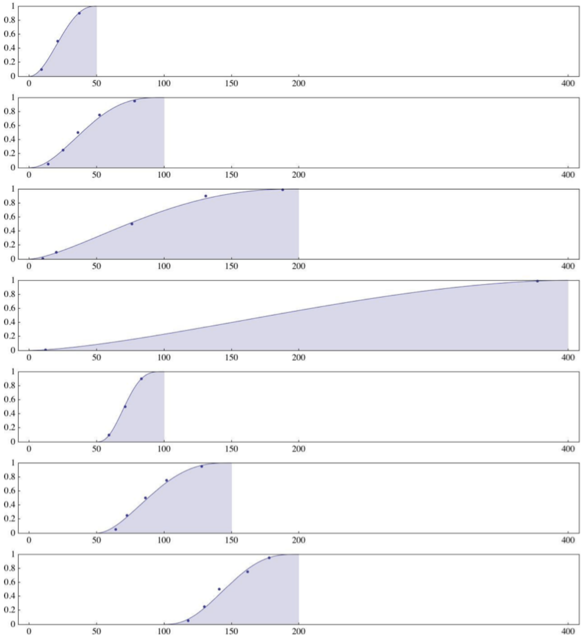
xl	xu	p	CE
0	50	0.00	0
0	50	0.10	9.0
0	50	0.50	21.0
0	50	0.90	37.0
0	50	1.00	50.0

•
•
•

xl	xu	p	CE
100	200	0.00	100.0
100	200	0.05	118.0
100	200	0.25	130.0
100	200	0.50	141.0
100	200	0.75	162.0
100	200	0.95	178.0
100	200	1.00	200.0

- Fix lottery range $[x_l, x_u]$
- Assign $u_{[x_l,x_u]}(x_l) = 0$ and $u_{[x_l,x_u]}(x_u) = 1$
- Following the vNM idea: $u_{[x_l,x_u]}(CE) = p$
- We fit a nonlinear function $u_{[x_l,x_u]} : [x_l, x_u] \to [0,1]$ with two restrictions given above

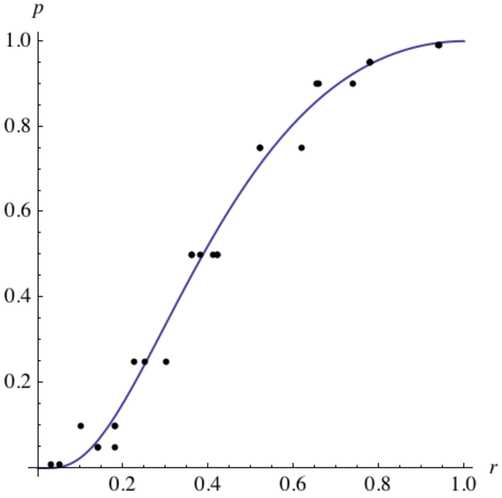
Fitting range-dependent utility functions



- Conceptually interesting
- Operationally demanding:
  - Eliciting different utility funtions for different lottery ranges

Fitting the decision utility function

- Observation
  - Range-dependent utilities differ mostly in **stretch and shift** of lottery consequences
- Normalize all lottery ranges and consequences into a common interval $[0, 1]$
- Define a single function $D : [0, 1] \to [0, 1]$, called the **decision utility function**
- Fit the function with the data



Setup

- X – set of monetary alternatives
- L – set of finite support **lotteries** P
  - L^d – set of degenerate lotteries P^x
- standard mixing operation:
  $(\alpha P + (1-\alpha)Q)(x) = \alpha P(x) + (1-\alpha)Q(x)$
- **lottery range** $\mathrm{Conv}(\mathrm{supp} P)$
- $L^c_{[x_l, x_u]}$ – set of **lotteries comparable within range** $[x_l, x_u]$ is the union of two sets:
  - $L_{[x_l, x_u]}$ – set of lotteries with range equal to $[x_l, x_u]$
  - $L^d_{[x_l, x_u]}$ – set of degenerate lotteries with support in $[x_l, x_u]$

Axioms

A "range-dependent" preference relation $\succsim \subset L \times L$ satisfies the following axioms:

Axiom (1)

Weak Order: *$\succsim$ is complete and transitive.*

Axiom (2)

Within-Range Continuity: *For any interval $[x_l, x_u] \subset X$, $x_l < x_u$ and for every $Q \in L^c_{[x_l, x_u]}$ the following holds:*

$$P^{x_u} \succ Q \succ P^{x_l} \implies$$
$$\exists \alpha, \beta \in (0,1) : \alpha P^{x_u} + (1-\alpha)P^{x_l} \succ Q \succ \beta P^{x_u} + (1-\beta)P^{x_l}.$$

Axioms

Axiom (3)

Within-Range Independence: *For any interval* $[x_l, x_u] \subset X$, $x_l < x_u$, *for every* $P, Q, R \in L$, *such that* $\alpha P + (1-\alpha)R, \alpha Q + (1-\alpha)R \in L^c_{[x_l, x_u]}$, *for all* $\alpha \in (0, 1]$ *the following holds:*

$$P \succsim Q \iff \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R, \ \forall \alpha \in [0, 1].$$

Axiom (4)

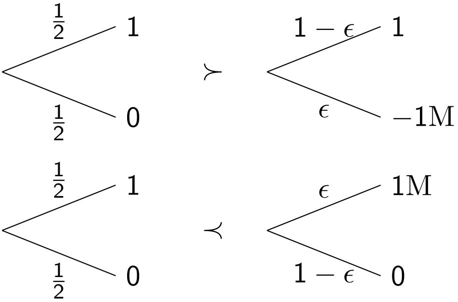
Monotonicity: *For all* $x, y \in X$ *the following holds:*

$$x > y \iff P^x \succ P^y$$

Discussion on axioms

Monotonicity means "more is better". Continuity and Independence required to hold only for lotteries comparable within the same range.

Within-Range Continuity: allows violations of continuity when lottery ranges differ



for all $\epsilon > 0$

for all $\epsilon > 0$

Discussion on axioms

Within-Range Independence: allows violations of independence when lottery ranges differ.

$$P \begin{cases} \frac{1}{2} & 50 \\ \frac{1}{2} & 40 \end{cases} \succ \; Q \begin{cases} \frac{1}{2} & 100 \\ \frac{1}{2} & 0 \end{cases}$$

$$\begin{cases} \frac{9}{10} & Q \\ \frac{1}{10} & -100 \end{cases} \succ \begin{cases} \frac{9}{10} & P \\ \frac{1}{10} & -100 \end{cases}$$

Range-dependent utility representation

Theorem (Range-dependent utility)

A preference relation $\succsim \subset L \times L$ satisfies axioms A1–A4 if and only if for every interval $[x_l, x_u] \subset X, x_l < x_u$ there exists a unique strictly increasing and surjective function $u_{[x_l, x_u]} : [x_l, x_u] \to [0, 1]$, such that for every pair of lotteries $P, Q \in L$ the following holds:

$$P \succsim Q \iff \mathrm{CE}(P) \geq \mathrm{CE}(Q), \tag{1}$$

where the certainty equivalent is defined as:

a) $\boxed{\mathrm{CE}(P) = u^{-1}_{\mathrm{Rng}(P)} \left[\sum_{x \in X} P(x) u_{\mathrm{Rng}(P)}(x)\right]}$ *for any* $P \in L \setminus L^d$,

b) $\mathrm{CE}(P^x) = x, x \in X$ *for any* $P^x \in L^d$.

The proof main idea

- Construct a **strictly increasing** and **surjective** mapping $u_{[x_l,x_u]} : [x_l, x_u] \to [0, 1]$.
- The **inverse** $u^{-1}_{[x_l,x_u]} : [0, 1] \to [x_l, x_u]$ exists
- So **CE** values are **well defined**
- Lotteries are **compared** on a monetary scale

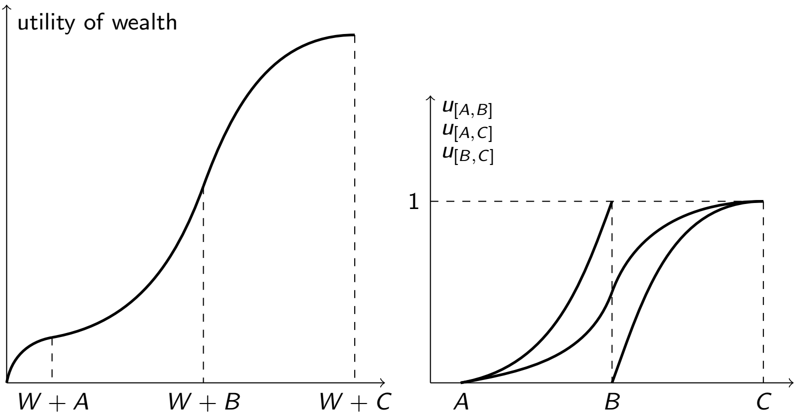
Intuition:

- The same consequence might be assigned two different utility values depending which lottery (with different ranges) it appears in.
- Hence CE values represent choices btw. lotteries with different ranges instead of utility values.

The intersection of range-dependent utility and EU

1. **The case of universal range**: In real life there always exists a tiny chance to die at once or to find a billion dollars on the street.
   - narrow framers (exhibiting EU paradoxes) and broad framers (rational)
2. **The case of consequentialism**: The family $(u_{[x_l,x_u]})$ is induced from u by taking: $u_{[x_l,x_u]}(x) = \frac{u(W+x)-u(W+x_l)}{u(W+x_u)-u(W+x_l)}$, $\forall x \in [x_l, x_u]$.

The case of consequentialism



Additional axiom: Shift and scale invariance

Definition

*For a lottery $P \in L$, $P : X \to [0,1]$ define its α, β-**transformation** $P_{\alpha,\beta} \in L$, $P_{\alpha,\beta} : X \to [0,1]$, such that $P(x) = P_{\alpha,\beta}(\alpha x + \beta)$, where $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$, $x \in X$ and $\alpha x + \beta \in X$, for all $x \in \mathrm{supp}(P)$.*

Axiom (5)

Scale and Shift invariance: *Let $P, Q \in L^c_{[x_l, x_u]}$ for some $[x_l, x_u] \subset X$, $x_l < x_u$. Then the following holds:*
$P \succsim Q$ iff $P_{\alpha,\beta} \succsim Q_{\alpha,\beta}$, for any
$\alpha > 0, \beta \in \mathbb{R}$: $P_{\alpha,\beta}, Q_{\alpha,\beta} \in L^c_{\alpha x_l + \beta, \alpha x_u + \beta}$.

In what follows it is assumed that $[0,1] \subset X$.

The decision utility representation

Theorem (Decision utility)

A preference relation $\succsim \subset L \times L$ *satisfies axioms A1–A5 if and only if there exists a unique strictly increasing and surjective function* $D : [0,1] \to [0,1]$, *such that for every pair of lotteries* $P, Q \in L$ *the following holds:*

$$P \succsim Q \iff \mathrm{CE}(P) \geq \mathrm{CE}(Q), \tag{2}$$

where the certainty equivalent is defined as:

a) $\boxed{\mathrm{CE}(P) = x_l + (x_u - x_l) D^{-1} \left[\sum_{x \in X} P(x) D \left(\frac{x - x_l}{x_u - x_l} \right) \right]}$, *for any* $P \in L \setminus L^d$, *where* $x_l = \min(\mathrm{Rng}(R))$, $x_u = \max(\mathrm{Rng}(R))$,

b) $\mathrm{CE}(P^x) = x, x \in X$ *for any* $P^x \in L^d$.

Discussion on the axiom

1. The family $(u_{[x_l,x_u]})$ is induced from a single decision utility function D by taking:

$$u_{[x_l,x_u]}(x) := D\left(\frac{x - x_l}{x_u - x_l}\right), \ \forall x \in [x_l, x_u].$$

2. Axiom (5) together with range-dependence reminds of **Parducci's range principle**
3. Due to this axiom the model exhibits **Constant Risk Aversion** of Safra and Segal (1998)
   - The model intersects EU in the case of risk neutrality
     - shift invariance equivalent to CARA, scale invariance equivalent to CRRA, both equivalent to an affine utility

Observational equivalence btw. Decision Utility and Dual Theory

Consider a binary lottery payoff $(x_l, 1-p; x_u, p)$

$$\textbf{Decision utility: } \mathrm{CE}(\mathbf{x}) = x_l + (x_u - x_l) D^{-1}(p),$$
$$\textbf{Dual Theory: } \mathrm{CE}(\mathbf{x}) = x_l + (x_u - x_l) w(p).$$

- The same predictions iff $D^{-1}(p) = w(p)$ for every $p \in [0,1]$.
- Evidence for binary lottery provides equal support for probability weighting and DU.
- For more than 2 outcomes the models can be discriminated.

Monotonicity and continuity

Definition

The CE **functional is monotonic wrt FOSD** *if whenever* $\mathbf{x} \succ_{FOSD} \mathbf{y}$, *then* $\mathrm{CE}(\mathbf{x}) > \mathrm{CE}(\mathbf{y})$.

Definition

The CE **functional is continuous** *if for every sequence of lottery payoffs* $\{\mathbf{x}_n\}$, *where* $n \in \mathbb{N}$ *and each* $\mathbf{x}_n$ *is distributed according to* F_n, *converging in distribution to the lottery payoff* $\mathbf{y}$ *distributed according to* G, *the following holds:* $\lim_{n\to\infty} \mathrm{CE}(\mathbf{x}_n) = \mathrm{CE}(\mathbf{y})$.

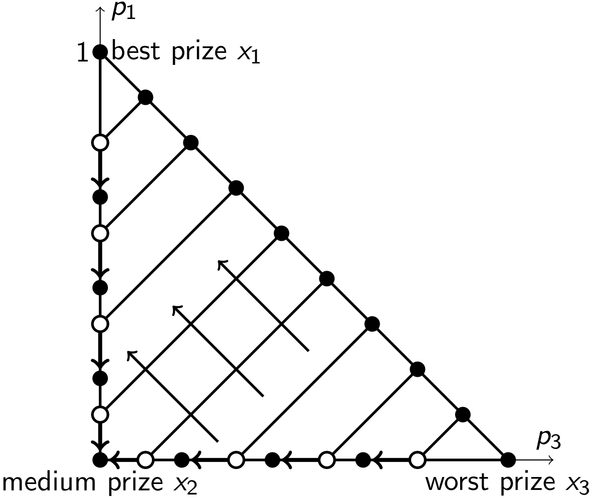
Monotonicity and continuity in the decision utility model

Define: $C(\eta) = 1 - D(1-\eta)$, $\eta \in [0,1]$. And then also $\mathrm{RRA}_D(\eta) = -\frac{\eta D''(\eta)}{D'(\eta)}$, $\mathrm{RRA}_C(\eta) = -\frac{\eta C''(\eta)}{C'(\eta)}$.

Theorem (Monotonicity and Continuity)

1) *The CE* **functional is monotonic wrt FOSD** *if and only if* RRA_D *and* RRA_C *are non-decreasing for all* $\eta \in [0,1]$
2) *The CE* **functional is continuous** *if and only if D is linear.*
   a) *Continuity wrt. upper range increase holds if and only if* RRA_D *is constant (power function).*
   b) *Continuity wrt. lower range increase holds if and only if* RRA_C *is constant (inverse power function).*

Indifference lines for the decision utility satisfying monotonicity



Example 1: The CDF of the Beta distribution

$$D(x) = A \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt,$$

where $x \in [0,1]$, $A = \frac{1}{\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt}$, and $\alpha, \beta > 0$.

Monotonicity conditions are satisfied in four special cases:

a) **linear**: $D(x) = x$, $\alpha = \beta = 1$,
b) **concave inverse power**: $D(x) = 1 - (1-x)^{\beta}, \beta > 1$, $\alpha = 1$,
c) **convex power**: $D(x) = x^{\alpha}, \alpha > 1$, $\beta = 1$,
d) **all S-shaped functions in this family**, $\alpha, \beta > 1$.

Example 2: The CDF of the Two-Sided Power Distribution

$$D(x) = \begin{cases} x_0 \left(\frac{x}{x_0}\right)^{\alpha}, & 0 \le x \le x_0, \\ 1 - (1 - x_0)\left(\frac{1-x}{1-x_0}\right)^{\alpha}, & x_0 \le x \le 1, \end{cases}$$

where $x_0 \in (0,1)$, $\alpha > 0$.

Monotonicity conditions are satisfied in four special cases:

a) **linear**: $D(x) = x$, $\alpha = 1$,

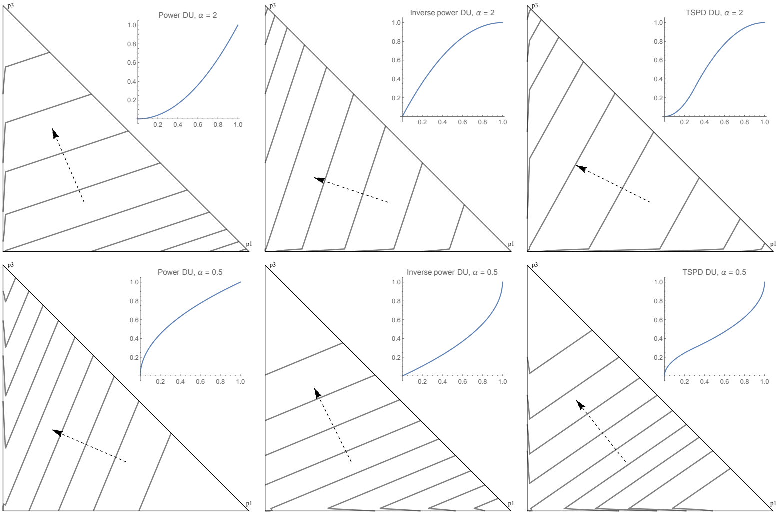
b) **concave inverse power**: $D(x) = 1 - (1-x)^{\alpha}, \alpha > 1$, $x_0 = 0$,

c) **convex power**: $D(x) = x^{\alpha}, \alpha > 1, x_0 = 1$,

d) **all S-shaped functions in this class**, $\alpha > 1$, $x_0 \in (0,1)$.

All inverse S-shaped functions in both classes are excluded.

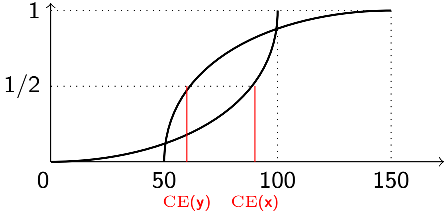
Indifference lines for TSPD decision utilities



Example: Monotonicity violation in the RDU model

Consider two lotteries with different ranges:

- $\mathbf{x} = (0, \frac{1}{2}; 100, \frac{1}{2})$
- $\mathbf{y} = (50, \frac{1}{2}; 150, \frac{1}{2})$

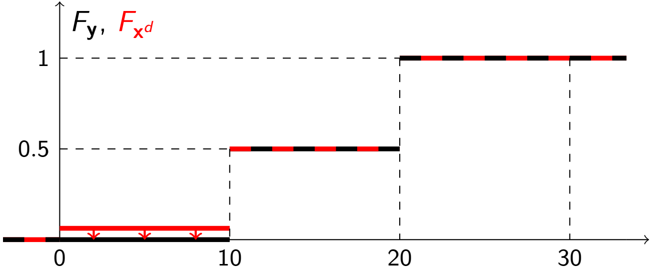


- Axiom A5 in the DU model imposes restrictions.
- But it is not enough to ensure monotonicity wrt FOSD — see below

Downward range change

$$\mathbf{y} = \left(10, \tfrac{1}{2}; 20, \tfrac{1}{2}\right)$$

$$\mathbf{x}^d = \left(0, \epsilon; 10, \tfrac{1}{2} - \epsilon; 20, \tfrac{1}{2}\right) \;\; \epsilon > 0$$



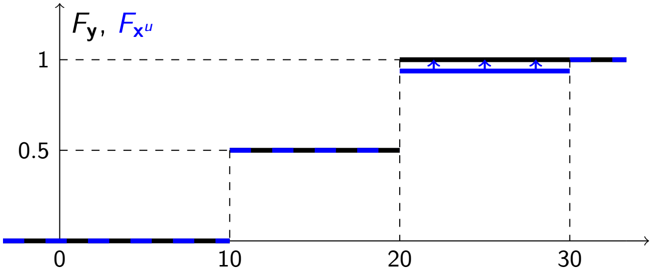
$\mathbf{x}^d \prec_{\text{FOSD}} \mathbf{y}$. Monotonicity requires $\lim_{\epsilon \to 0^+} \mathrm{CE}(\mathbf{x}^d) \leq \mathrm{CE}(\mathbf{y})$.

$\mathbf{x}^d \xrightarrow{D} \mathbf{y}$. Continuity requires $\lim_{\epsilon \to 0^+} \mathrm{CE}(\mathbf{x}^d) = \mathrm{CE}(\mathbf{y})$.

Upward range change

$$\mathbf{y} = \left(10, \tfrac{1}{2}; 20, \tfrac{1}{2}\right)$$

$$\mathbf{x}^u = \left(10, \tfrac{1}{2}; 20, \tfrac{1}{2} - \epsilon; 30, \epsilon\right), \ \epsilon > 0$$

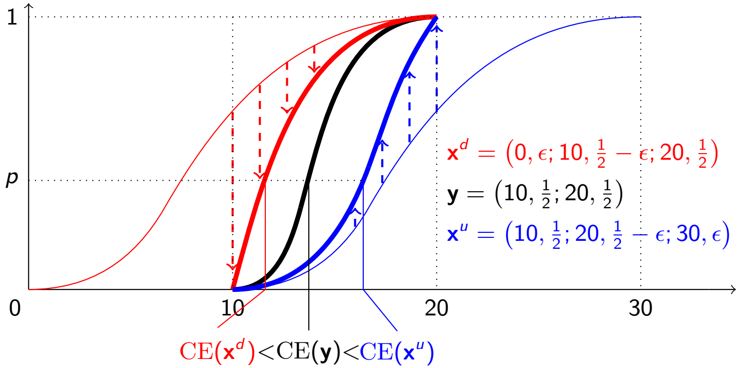


$\mathbf{x}^u \succ_{\text{FOSD}} \mathbf{y}$. Monotonicity requires $\lim_{\epsilon \to 0^+} \mathrm{CE}(\mathbf{x}^u) \geq \mathrm{CE}(\mathbf{y})$.

$\mathbf{x}^u \xrightarrow{D} \mathbf{y}$. Continuity requires $\lim_{\epsilon \to 0^+} \mathrm{CE}(\mathbf{x}^u) = \mathrm{CE}(\mathbf{y})$.

Monotonicity and continuity for S-shaped functions

From now on let $\mathrm{CE}(\mathbf{x}^d)$, $\mathrm{CE}(\mathbf{x}^u)$ denote the limits as $\epsilon \to 0^+$.



- **Continuity is generally violated** in the decision utility model
- Monotonicity is **typically satisfied for S-shaped** fcns
- Monotonicity is **always violated for inverse S-shaped** fcns

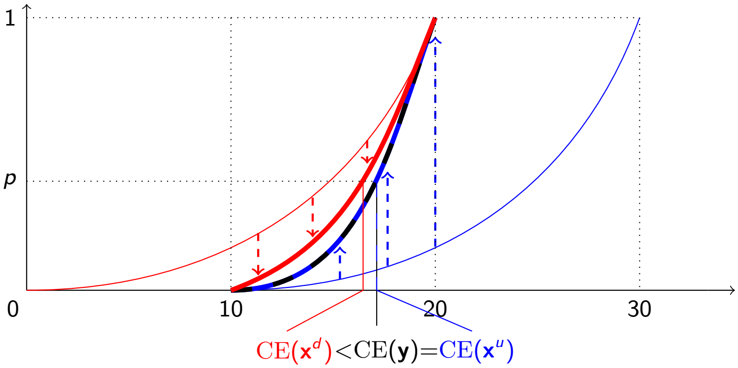
Monotonicity and continuity for the limiting functions

limiting functions	$D(x)$	$\mathrm{CE}(\mathbf{x}^d)$	$\mathrm{CE}(\mathbf{y})$	$\mathrm{CE}(\mathbf{x}^u)$
convex power	x^2	**15.81**	17.07	17.07
concave power	$\sqrt{x}$	**14.57**	12.5	12.5
convex inverse power	$1-\sqrt{1-x}$	17.5	17.5	**15.43**
concave inverse power	$1-(1-x)^2$	12.93	12.93	**14.81**

- Power is continuous wrt upward range changes
- Inverse power is continuous wrt downward range changes
- Concave power and convex inverse power violate monotonicity
- Convex power and concave inverse power satisfy monotonicity

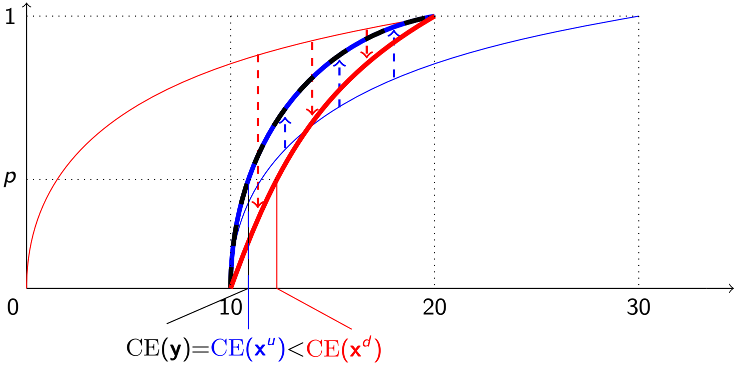
Monotonicity for convex power function

Consider $\boxed{D(x) = x^2}$. Intuition: It becomes less and less curved.



Monotonicity violation for concave power function

Consider $\boxed{D(x) = \sqrt{x}}$. Intuition: It becomes less and less curved.



EU Paradoxes

Coexistence of gambling and insurance:

$$\begin{aligned}(P - pP, p; -pP, 1 - p) &\succ (0, 1),\\ (H, 1 - p; 0, p) &\prec (H - pH, 1).\end{aligned}$$

This pattern of preferences is predicted by the decision utility model if the following conditions are satisfied:

$$\boxed{p > \max(D(p), 1 - D(1 - p))}$$

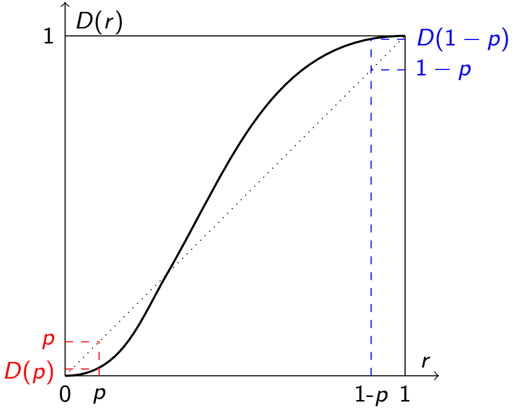


Figure: gambling – no gambling and insurance – no insurance comparison.

- binary lotteries: DU is observationally equivalent to DT
- However psychologically very different, based on an S-shaped utility function and hence much closer to Markowitz (1952)

Russian roulette

Two situations:

1. A six-shooter with 4 loaded chambers. How much would you pay to remove one bullet?
2. A six-shooter with 2 loaded chambers. How much would you pay to remove two bullets?

Expected Utility Theory predicts that the two prices should be the same (Assumption: if you die you don't care)

$$\begin{aligned}
\frac{4}{6}u(\mathrm{death}) + \frac{2}{6}u(w) &= \frac{3}{6}u(\mathrm{death}) + \frac{3}{6}u(w-P) \\
\frac{2}{6}u(\mathrm{death}) + \frac{4}{6}u(w) &= u(w-Q)
\end{aligned}$$

Assuming that $u(\mathrm{death}) = 0$ and $u(w) = 1$, we get:

$$u(w-P) = 2/3 = u(w-Q) \quad \Rightarrow \quad P = Q$$

Russian roulette

Let's see how it is with the Decision Utility model:

$$\begin{aligned}
\mathrm{death} + (w - \mathrm{death}) D^{-1}\left(\frac{1}{3}\right) &= \mathrm{death} + (w - P - \mathrm{death}) D^{-1}\left(\frac{1}{2}\right) \\
\mathrm{death} + (w - \mathrm{death}) D^{-1}\left(\frac{2}{3}\right) &= w - Q
\end{aligned}$$

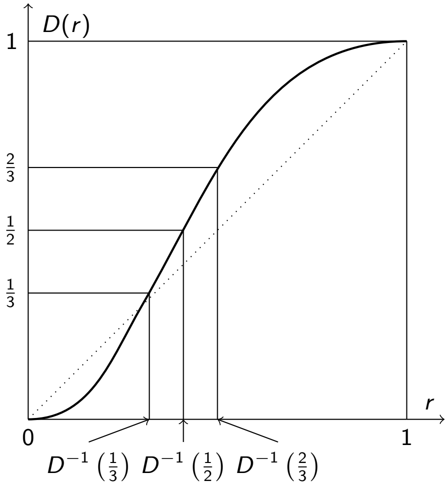
Hence we get the following conditions:

$$\begin{aligned}
\frac{D^{-1}\left(\frac{1}{3}\right)}{D^{-1}\left(\frac{1}{2}\right)} &= \frac{w - P - \mathrm{death}}{w - \mathrm{death}} \\
\frac{D^{-1}\left(\frac{2}{3}\right)}{D^{-1}(1)} &= \frac{w - Q - \mathrm{death}}{w - \mathrm{death}}
\end{aligned}$$

Finally we get:

$$Q > P \iff \frac{D^{-1}\left(\frac{2}{3}\right)}{D^{-1}(1)} < \frac{D^{-1}\left(\frac{1}{3}\right)}{D^{-1}\left(\frac{1}{2}\right)}$$

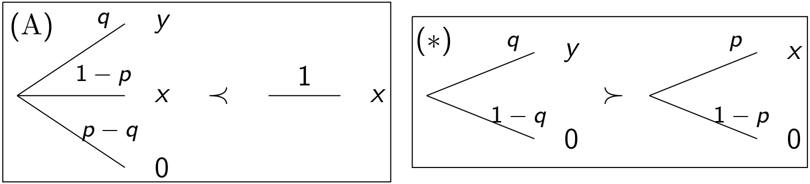
Russian roulette



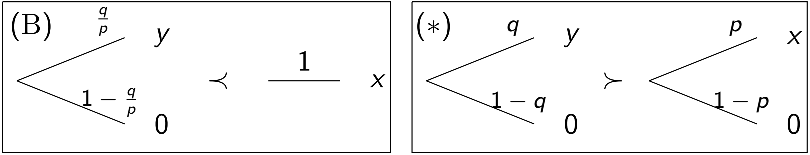
The Allais paradox and the Common Ratio effect

Let $y > x > 0$, $1 > p > q > 0$, $\frac{q}{p} > p$,
e.g. $y = \$4\ 000$, $x = \$3\ 000$, $q = 0.2$, $p = 0.25$.

Allais



CR



EU: (A),(B) equivalent and cannot coexist with (∗).
DU: (A),(B) equivalent and can coexist with (∗).
Rank: (A),(B) not equivalent and can coexist with (∗).

The Allais paradox and the Common Ratio effect

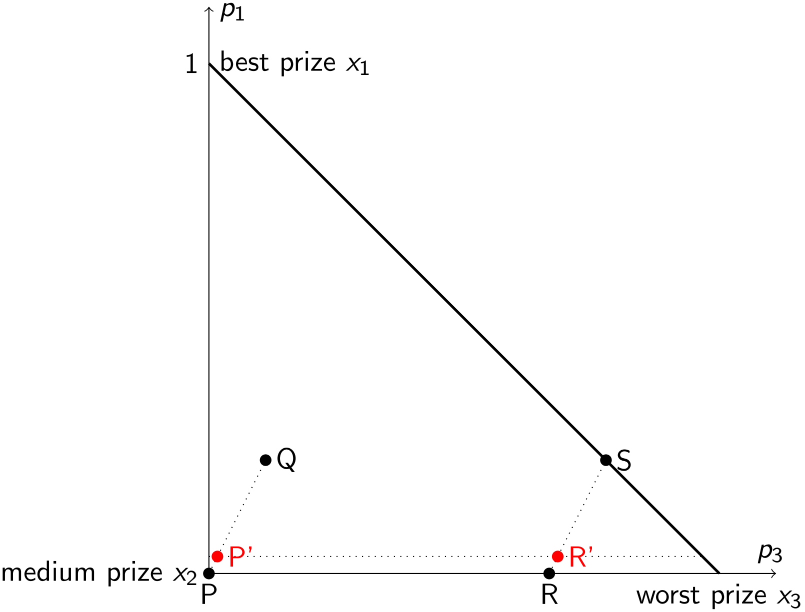
EU: $\underbrace{\frac{u(W+x)}{u(W+y)} < \frac{q}{p}}_{\text{(A),(B)}} \overbrace{< \frac{u(W+x)}{u(W+y)}}^{(*)}$ ... contradiction

DU: $\underbrace{D^{-1}\left(\frac{q}{p}\right) < \frac{x}{y}}_{\text{(A),(B)}} \overbrace{< \frac{D^{-1}(q)}{D^{-1}(p)}}^{(*)}$ ... satisfied when D is flat in the upper and steep in the middle part of its domain.

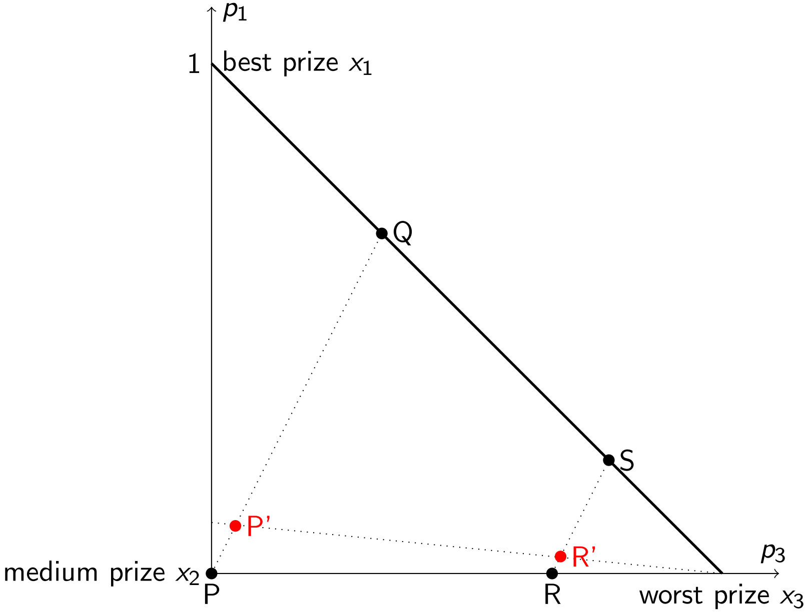
Yaari: $\underbrace{\frac{w(q)}{w(q)+1-w(1-p+q)} < \frac{x}{y}}_{\text{(A)}} \overbrace{< \frac{w(q)}{w(p)}}^{(*)}$

$$\underbrace{w\left(\frac{q}{p}\right) < \frac{x}{y}}_{\text{(B)}} \overbrace{< \frac{w(q)}{w(p)}}^{(*)}$$

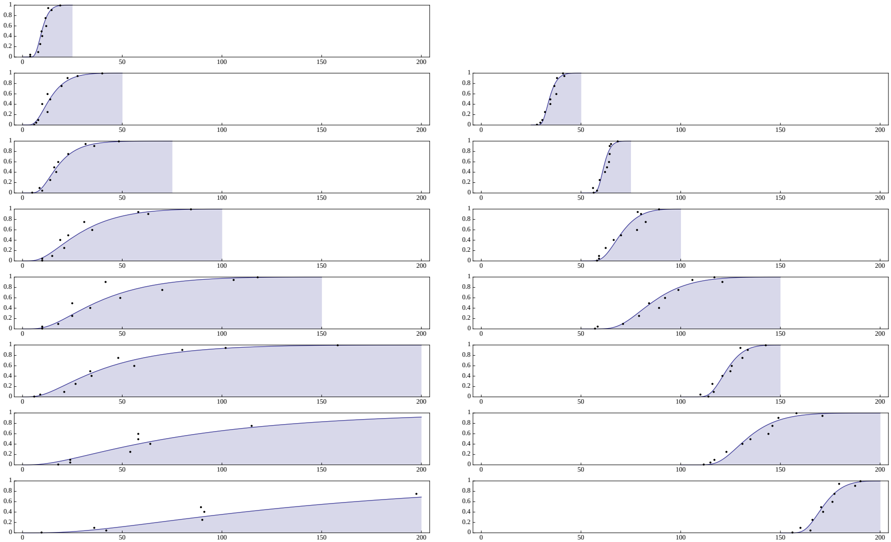
The Allais lotteries in the Marschak-Machina triangle



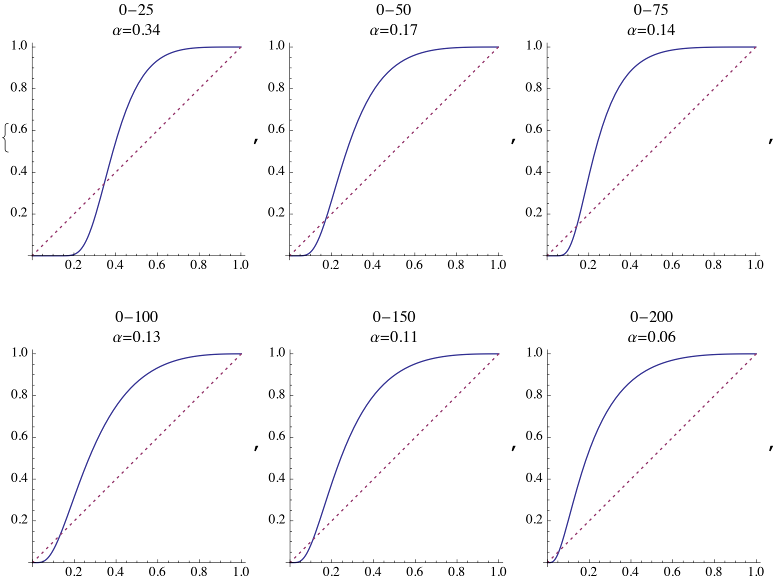
The CR lotteries in the Marschak-Machina triangle



Extension I: Wealth effects present in the Gonzales, Wu (1999) data



The decision utility function is not constant



Aspiration level

Risk setup: Consider the set of binary lotteries $\mathcal{L}^{bin}_{[x_l,x_u]}$ with range $[x_l, x_u]$: $\mathbf{x}^p = (x_u, p; x_l, 1-p), p \in [0,1]\}$
Let's focus on the S-shaped decision utility function.

Definition

The **relative aspiration level** *is the value* $al \in [0,1]$ *such that for* $\mathbf{x}^{al} \in \mathcal{L}^{bin}_{[x_l,x_u]}$*:*

$$CE(\mathbf{x}^{al}) = \mathrm{E}[\mathbf{x}^{al}]$$

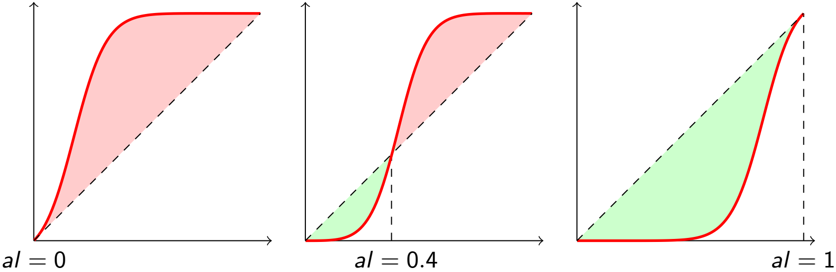
And additionally:

$$\begin{aligned} \forall p \in [0,1] : p < al, \quad & CE(\mathbf{x}^p) > \mathrm{E}[\mathbf{x}^p] \qquad \mathrm{risk-loving} \\ \forall p \in [0,1] : p > al, \quad & CE(\mathbf{x}^p) < \mathrm{E}[\mathbf{x}^p] \qquad \mathrm{risk-aversion} \end{aligned}$$

Moreover, the value $CE(\mathbf{x}^{al})$ *is called the* **nominal aspiration level** *and is denoted by* AL*.*

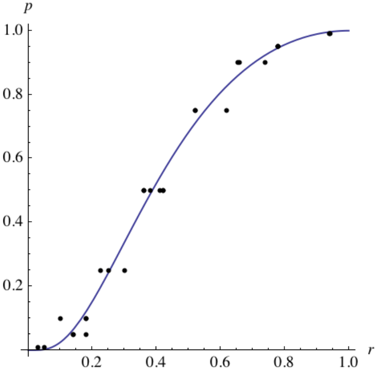
Aspiration levels and risk attitudes

Risk loving occurs $\forall \mathbf{x} : CE(\mathbf{x}) < AL$ (until we reach AL)
Risk aversion occurs $\forall \mathbf{x} : CE(\mathbf{x}) > AL$ (after we reach AL)



Weak wealth effects in TK 1992 data

In the decision utility model the **relative aspiration level is constant** for all lottery ranges: $al = \lambda = \text{const}$



Two models satisfying Axiom 4"

Depending on the RRA parameter of u, we will get different models. For example

- $\boxed{u(x) = \log(x)}$ (RRA=1)

$$\begin{aligned} \lambda &= \frac{\log(W + AL) - \log(W + x_l)}{\log(W + x_u) - \log(W + x_l)} \\ W + AL &= (W + x_l)^{1-\lambda}(W + x_u)^{\lambda} \end{aligned}$$

Hence $\boxed{W + AL = \mathbb{G}[(W + x_l, 1 - \lambda; W + x_u, \lambda)]}$

- $\boxed{u(x) = 1 - 1/x}$ (RRA=2)

$$\begin{aligned} \lambda &= \frac{-1/(W + AL) + 1/(W + x_l)}{-1/(W + x_u) + 1/(W + x_l)} \\ \frac{1}{W + AL} &= (1 - \lambda)\frac{1}{W + x_l} + \lambda\frac{1}{W + x_u} \end{aligned}$$

Hence $\boxed{W + AL = \mathbb{H}[(W + x_l, 1 - \lambda; W + x_u, \lambda)]}$

- The second model fits the data best

The simple model is consistent with the stylized facts

$$\frac{1}{W+AL} = (1-\lambda)\frac{1}{W+x_l} + \lambda\frac{1}{W+x_u}$$

$$al = \frac{\lambda(W+x_l)}{\lambda(W+x_l)+(1-\lambda)(W+x_u)}$$

- The **relative aspiration level** al:
  - increases with wealth
  - converges to the decision utility model as wealth goes to infinity
  - decreases with x_u
  - increases with x_l
- The **nominal aspiration level** AL:
  - is bounded
  - tends to zero as wealth goes to zero (bankruptcy)

Having determined al, we fit a two-sided power distribution. In case of the Gonzales, Wu (1999) data SSE in our model is **74.5** and in CPT it is **98.3**

Extension II: Range-dependent utility under ambiguity

- **Risk:** $\mathbf{x} = (x_1, p_1; ...; x_n, p_n)$, where p_i are probabilities
- Uncertainty case $\mathbf{x} = (x_1, E_1; ...x_n, E_n)$, where E_i are events
  - **Full uncertainty**: no information about probability
  - **Ambiguity**: partial information about the probability

Let's denote by $\mathcal{A}$ as the set of all acts with finite set of events.

Literature

0) Full uncertainty:

- Wald criterion: $\min_{p \in \Delta} \mathbb{E}_p u(\mathbf{x})$
- Hurwicz criterion: $\alpha \min_{p \in \Delta} \mathbb{E}_p u(\mathbf{x}) + (1-\alpha) \max_{p \in \Delta} \mathbb{E}_p u(\mathbf{x})$

1) Ambiguity: Multiple-priors models:

- Subjective beliefs:
  - Gilboa, Schmeidler (1989): $\min_{p \in C} \mathbb{E}_p u(\mathbf{x})$
  - Ghirardato, Maccheroni, Marinacci (2004): $\alpha \min_{p \in C} \mathbb{E}_p u(\mathbf{x}) + (1-\alpha) \max_{p \in C} \mathbb{E}_p u(\mathbf{x})$
- Objective but imprecise probability:
  - Jaffray (1989): EU if probabilities belong to intervals $\alpha \min_{p \in \mathcal{P}} \mathbb{E}_p u(\mathbf{x}) + (1-\alpha) \max_{p \in \mathcal{P}} \mathbb{E}_p u(\mathbf{x})$
  - Gajdos, Hayashi, Tallon, Vergnaud (2008): contraction model $\min_{p \in \Phi(\mathcal{P})} \mathbb{E}_p u(\mathbf{x})$, where Φ transforms objective info into subjective beliefs

Literature

2) Ambiguity: Second-order beliefs

- Klibanoff, Marinacci, Mukerji (2005): $\mathbb{E}_{\mu}\Phi(\mathbb{E}_{p}u(\mathbf{x}))$,
  - μ is second-order probability
  - Φ is the ambiguity attitude function

3) Ambiguity: source dependence

- Chew, Sagi (2006), (2008): source dependence, small worlds
- Ergin, Gul (2009): source dependence linked with second-order beliefs

Our aim

In our model we aim to incorporate:

- Imprecise objective info + subjective beliefs
  - Even with no information people state CEs
  - Objective information should matter
- Second-order probability
  - But objective, not subjective
- Source-depedence
  - Different utility functions for different uncertainty levels

Hurwicz criterion is range-dependent

Definition (Hurwicz criterion)

In the complete ignorance case, given the pessimism index $1-\lambda$ *evaluate a given act* $\mathbf{x} \in \mathcal{A}$ *with the following criterion:*

$$(1-\lambda)x_l + \lambda x_u$$

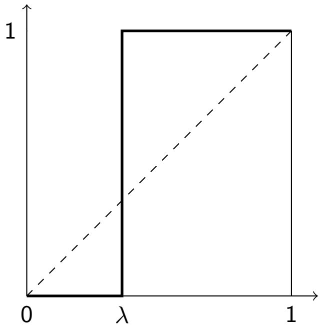
We can translate this criterion into the decision utility framework. Let's define the following decision utility correspondence:

$$D(x) = \begin{cases} 0 & \text{for } x \in [0,\lambda) \\ [0,1] & \text{for } x = \lambda \\ 1 & \text{for } x \in (\lambda, 1] \end{cases} \tag{3}$$

This correspondence is then used to obtain the certainty equivalent for an act $\mathbf{x} \in \mathcal{A}$:

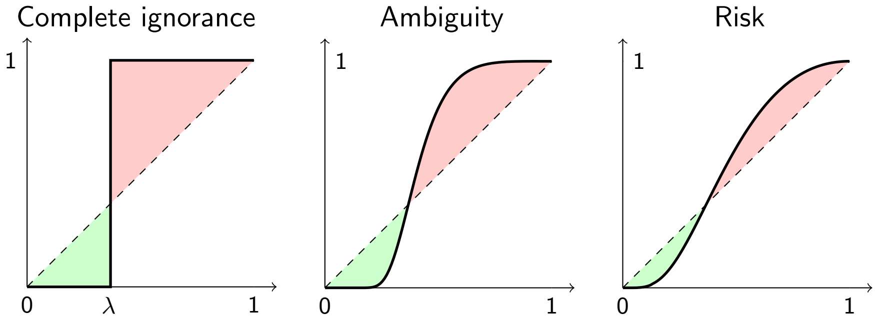
$$CE(\mathbf{x}) = D^{-1}(p) = \lambda, \ \forall p \in (0,1)$$

The decision utility function for the complete ignorance case: the Hurwicz criterion



- We treat this as a limiting case in which uncertainty level is maximal
- After renormalizing it is a range-dependent version of a satisficing utility of Simon (1956) with aspiration level λ.

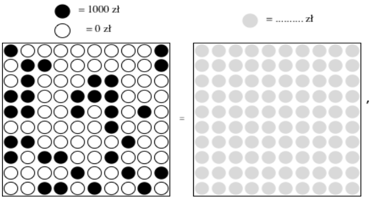
Ignorance, Ambiguity and Risk: changing the function's slope



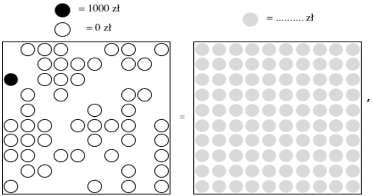
How to measure and simulate ambiguity? - The experiment

Sample problems from the experiment

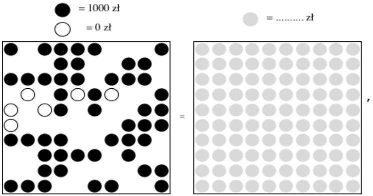
Sample risk problem:



Sample uncertainty problem:



Sample uncertainty problem:



Ellsberg paradox in the decision utility model

- Increasing Slope = Increasing Uncertainty
- It works well for Ellsberg: