Range-Dependent Utility

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Abstract

First, this paper introduces and axiomatizes range-dependent utility as a new conceptual framework for decision-making under risk. It is a simple and well-defined generalization of Expected Utility Theory in which utility depends on the range of lottery outcomes. Second, a special case of this framework is proposed for prediction. It is based on applying a single utility function (decision utility) to every normalized lottery range. The resulting decision utility model predicts well-known Expected Utility paradoxes without recourse to probability weighting. Necessary and sufficient conditions for the model to satisfy monotonicity with respect to FOSD are identified. The typical decision utility function is S-shaped which is confirmed both by experimental data and normative considerations.

Keywords: Range-Frequency model, Expected Utility, Certainty Equivalent, Allais paradox, probability weighting, stochastic dominance violations

JEL Classification Numbers: D81, D03, C91

1 Introduction

Many restaurants use extremely expensive dishes as decoys. One probably will not buy them, but will find other dishes a little cheaper. This example suggests that the attractiveness of a given price depends on the range of prices in which it occurs. (Parducci, 1995, p.31) gives another example of range-dependent judgments: *On tropical islands where the temperature is almost always in the 80s, the natives are sensitive to differences that seem hardly noticeable to us; thus, they complain of the extremes, of the heat when the temperature is in the high 80s, of the cold when it is in the low 80s*.

We believe that range effects are present also in the context of decisions under risk. What makes people buy the US Powerball lottery tickets, even though their expected value is usually well below their purchase price? Probably not the chance of winning one of the middle prizes, although winning something is always pleasant. Definitely not the tiny chance of matching all the numbers, although

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people know it is possible (you yourself might have seen a jackpot winner on TV recently). We believe it is the jackpot value that is the main feature that attracts people to playing Powerball. A $2-ticket buys the possibility to dream about being really rich. It is therefore the range of outcomes that makes the lottery so attractive. Clearly, the bigger the jackpot, the more tickets are sold. This is especially true when the quoted jackpot is larger than any previous one. Buying frenzies usually develop in such cases and the increase in ticket sales often exceeds the increase in jackpot value.[1] Changes in the lottery rules that went into effect for the Oct. 7, 2015 game only strengthened the effect described. Under the new rules the odds of winning the jackpot of a single ticket decreased from about a 1 in 175.2 million chance to 1 in 292.2 million. This should have decreased the popularity of the lottery, but instead it was meant to and succeeded in fostering ticket sales. Why? Because decreasing the probability of hitting the jackpot increased the likelihood that the jackpot would reach higher values[2], which in turn attracted more sales.

Range effects may work in the opposite direction as well. One of the authors of this paper was recently buying a vacation package. A beautiful destination suggested by the travel agent was, however, immediately rejected because of a possibility (although very tiny) of a terrorist attack. A similar reaction is observed when talking about monetary consequences. Many people decide not to enter a promising business or investment because of a tiny chance of losing a big amount of money or even become bankrupt; this happens no matter how attractive the other consequences are. The first question when faced with a risky prospect is often How much is to lose? and the maximum possible loss, rather than its exact likelihood, is the main factor impacting peoples decisions. These examples suggest that the upper and lower bounds of the payoff range play an important role in decisions involving risk and are often their main driver, especially since exact probabilities of events are usually unknown.

The goal of this paper is to apply the idea of range dependence to the context of decisions under risk. Two concepts our approach is based on are range effects in judgement and Expected Utility with narrow framing. After discussing them, we introduce our theory and discuss its main properties. All the proofs are relegated to Appendix 2.

2 Motivating concepts

2.1 Range effects in judgement

Range effects were first considered by Parducci (1965) in his Range-Frequency Theory. This theory describes psychophysical judgment as a compromise between two principles: the range principle and the frequency principle. The first one assumes that subjects locate each stimulus relative to the subjective end values. The place of a stimulus s_i in the range is reflected in the following definition:

$$R_i = \frac{s_i - s_{min}}{s_{max} - s_{min}},$$

with s_{min} and s_{max} representing the lowest and the highest of the stimulus values in the context of stimuli affecting the judgement of s_i. R_i is thus a proportion that can take any value between 0 and 1. On the other hand the frequency principle asserts that differences in response tend to be proportional to

[1]The expected ticket value may actually decrease due to the increased possibility of multiple winning tickets as was the case for the first billion dollar jackpot in mid January 2016 (`http://www.durangobill.com/PowerballOdds.html`).

[2]The chance of a billion-dollar jackpot in any five-year period rose more than seven-fold, from 8.5% to 63.4%, as calculated by Walt Hickey of `http://fivethirtyeight.com`.

differences in stimulus rank:

$$F_i = \frac{r_i - 1}{N - 1},$$

where r_i is the rank of stimulus i, and 1 and N are the ranks of the smallest and the largest of the stimulus values. The final judgment function is a weighted sum of the range and the frequency principle. Major part of Parduccis work was devoted to analyzing the stimulus distribution and noting that its skewness impacts peoples judgments. In the present work, however, we apply only the range principle. As shown in the paper, the range effects alone may capture the most important phenomena discussed in the literature on decision-making under risk. Therefore we regard the frequency principle as the second order effect.

Although Range-Frequency is the underlying theory, more recent studies provide a deeper explanation of adaptation to the stimulus range at the sensory level. The best example of neural adaptation is the ability of the eye to adjust to different ranges of light intensity. Webster et al. (2005) note that neurons have a limited dynamic range and in order to realize its full capacity a neuron's operating curve (of the sigmoidal shape) should be matched to the range of stimulus levels. Thus short ranges produce steep psychophysical functions, and wide ranges produce flatter functions (Lawless and Heymann, 1998). In his deliberations on biology, evolution and human nature, Robson (2002) states that utility that has a relative and local scale, rather than an absolute or global scale, may be biologically advantageous.

Adaptation to the stimulus range is observed also on a higher level of perception. For example Janiszewski and Lichtenstein (1999) postulate that a consumers assessment of the attractiveness of a market price depends on its comparison to the endpoints of the evoked price range, and show that prices for various foods (cereal, cookies, snacks, soup) are viewed as more (or less) expensive when less (or more) expensive products are added to the context. The same effect is observed for prices of airline tickets and 2-liter soda bottles (Niedrich et al., 2001). Thus increasing the upper bound of the price range for the fast-moving consumer goods produces a higher average price of goods selected by subjects (Bennett et al., 2003). Moreover, Yeung and Soman (2005) examine situations in which consumers choose between options that vary on two attributes and state that as their ranges widen, the range effect makes perceptual differences on both attributes look smaller. Cialdini (1993) argues that it is much more profitable for salespeople to present the expensive item first. After being exposed to the price of the large item (like a $495 suit), the price of the less expensive one (a $95 sweater) appears smaller by comparison. This effect is well described by the Weber Law, one of the fundamental laws of psychophysics. It states that the just noticeable difference is a constant proportion of the initial stimulus magnitude.

2.2 Expected Utility with narrow framing

The understanding of the concept of utility in the context of decision-making under risk has evolved since the introduction of Expected Utility Theory (von Neumann and Morgenstern, 1944). This theory is characterized by the set of axioms among which independence is a crucial one. It implies that the utility function used to evaluate lotteries is linear in probabilities. Expected Utility Theory does not assume any specific interpretation for the model parameters. The most common interpretation is that the decision maker's preferences over lotteries defined on wealth changes are induced from his preferences over lotteries defined on wealth levels. This interpretation, which Rubinstein (2012) calls the doctrine of consequentialism, together with the Expected Utility Hypothesis, is known as the Expected Utility of wealth model (Cox and Sadiraj, 2006; Palacios-Huerta and Serrano, 2006; Lewandowski, 2014).

This model was challenged from the outset in the persistent quest to reconcile the empirical evidence with the existing theories. Two approaches are generally taken. The first retains the Expected Utility Hypothesis but dispenses with the consequentialist interpretation. The second calls the entire model into question. A classic example of the first approach is provided by Markowitz (1952), who introduced the idea of reference dependence (utility defined over wealth changes) in place of the terminal wealth assumption to explain the coexistence of insurance and gambling. This idea was also one of the building blocks of Prospect Theory (Kahneman and Tversky, 1979), that assumes that gains and losses are perceived as monetary amounts relative to some reference point.

Reference dependence alone cannot, however, explain other well-known choice paradoxes, e.g. the Allais (1953) paradox. Kahneman and Tversky (1979) therefore included probability weighting, an idea initially developed by Edwards (1954). Probability weighting (in its original and later cumulative form) was subsequently adopted in many theories that attempted to explain empirical data (for a review see Starmer, 2000). Unlike the introduction of reference dependence, which departs only from the consequentialist interpretation, probability weighting departs from the Expected Utility hypothesis by violating the independence axiom.

We propose an alternative approach to reference dependence that retains linearity in probabilities. The assumption of consequentialism is replaced with range dependence. Instead of assuming a single reference point, we assume two naturally given reference points, viz. the minimum and the maximum lottery prize. These two reference points define the lottery range, relative to which each lottery outcome is evaluated. Our approach differs however from context-dependent modeling. We assume narrow framing: lottery valuation, which depends on its own range, is not influenced by other lotteries under consideration.

3 The idea of range-dependent utility

In their seminal contribution, von Neumann and Morgenstern (1944) proposed the following method of measuring preference for outcomes when gambles are involved. Suppose there are three outcomes, A, B, and C ranked in increasing order of preference. An individual is asked to choose between the following alternatives:

a) Receive C with probability p and A with probability $1-p$;

b) Receive B with probability 1.

If p is chosen so that the individual is indifferent between the two alternatives, and we assign a utility of 1 to C and a utility of 0 to A, then p is a measure of preference for B: $u(B) = pu(C) + (1-p)u(A) = p$, where u denotes the utility of outcomes. Note that the attractiveness of the middle outcome B is measured *relative* to that of outcomes A and C. Alternatively, given a lottery in a) with some probability p, one may choose an outcome B such that the individual is indifferent between receiving B or playing the lottery. We call B the certainty equivalent (CE) of the lottery.

3.1 Fitting range-dependent utilities

In what follows, we make use of the experimental data by Tversky and Kahneman (1992) that served them to derive Cumulative Prospect Theory parameters. We use the same data to motivate the range-dependent and the decision utility models. As this section aims only at presenting the idea, those estimation details that are unnecessary at this stage are discussed in Appendix 1.

The data report Certainty Equivalent values for 28 binary lotteries elicited from a group of subjects. There were 7 pairs of monetary outcomes: $(0, 50)$, $(0, 100)$, $(0, 200)$, $(0, 400)$, $(50, 100)$, $(50, 150)$, $(100, 200)$, and different probabilities of getting the higher outcome were considered for each of these pairs. The interval $[x_l, x_u]$, where x_l and x_u are the lower and the upper outcome, respectively, is

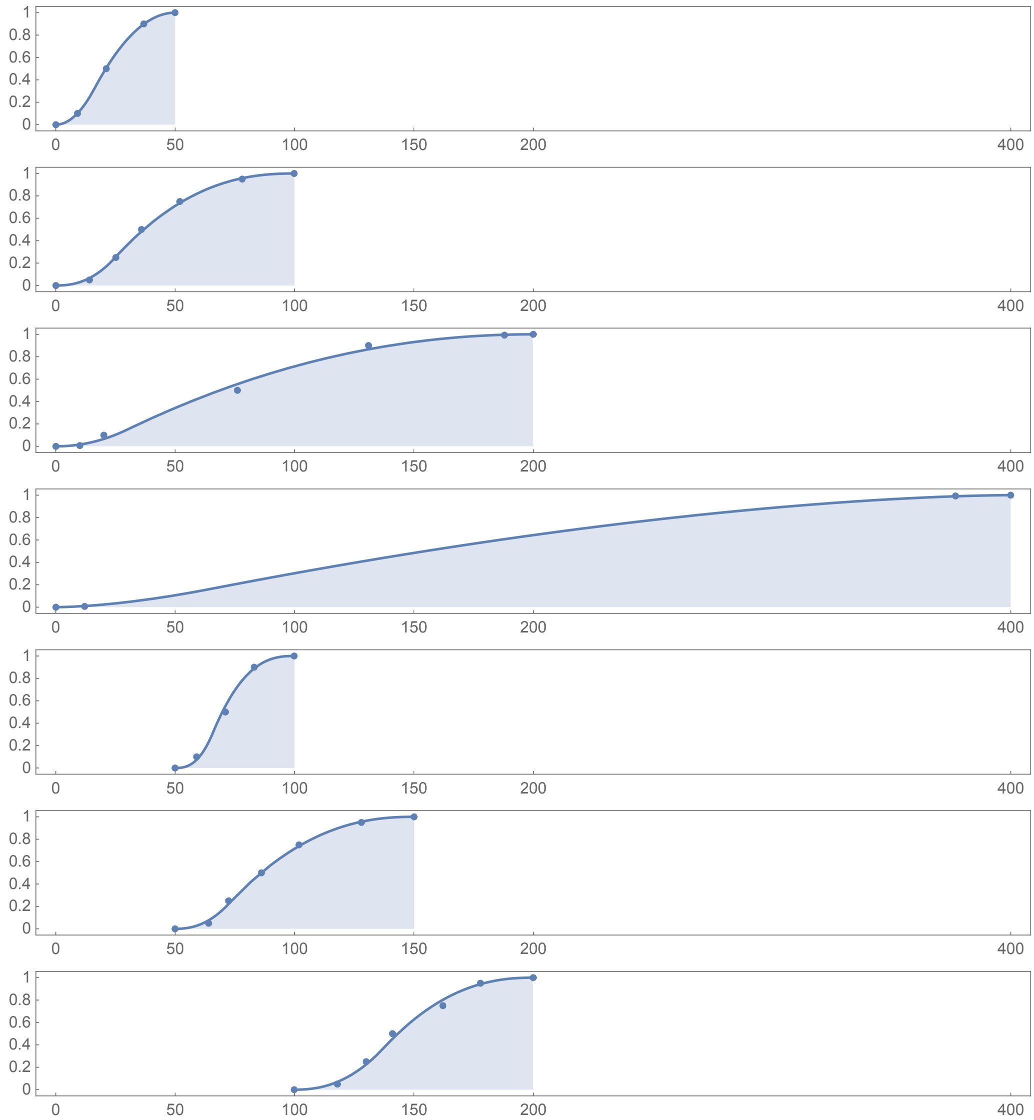

Figure 1: Range-dependent utilities fitted to Tversky and Kahneman (1992) data. The estimated models take the form of the shifted CDF of the Two-Sided Power Distribution (Kotz and van Dorp, 2004) with varying supports. These supports correspond to the respective lottery ranges: $[0, 50]$, $[0, 100]$, $[0, 200]$, $[0, 400]$, $[50, 100]$, $[50, 150]$, and $[100, 200]$. The CE values are measured on the horizontal axis and the probability of obtaining the greater of the two prizes on the vertical axis. For example the lottery $(\$0, 0.9; \$50, 0.1)$ has a CE value of \$9, which corresponds to the second from the left point in the upper graph.

referred to as the lottery range. We applied the von Neumann-Morgenstern idea of utility construction separately for lotteries having the same range. It follows that the CE utility value in a given range equals the probability of receiving the higher outcome. The (CE, probability) pairs were used to fit seven nonlinear models, separately for each lottery range. (We could as well just interpolate the data points over corresponding lottery ranges, but we wanted to obtain convenient parametric forms.) Two restrictions were imposed on the estimated functions: they should start at the point $(x_l, 0)$ and end at the point $(x_u, 1)$. These functions, which we call range-dependent utility functions,[3] are presented in Figure 1.

Observe that these functions are S-shaped for each range. Moreover, the function curvature at a given lottery outcome depends on the range in which this outcome occurs. For example, at the same value of \$75, the utility estimated in the range $[0, 100]$ is concave, the utility estimated in the range $[0, 200]$ is roughly linear, and the utility estimated in the range $[0, 400]$ is convex. This suggests that risk attitudes for a given monetary outcome depend on its relative position in the lottery range. More importantly, this finding is incompatible with the existence of a single utility function for all outcomes.

3.2 Fitting decision utility

Note that the range-dependent utility curves presented in Figure 1 have similar shapes. They differ mostly in the horizontal stretch and shift of lottery outcomes. Therefore we hypothesize that a single utility function defined on the interval [0,1] can be fitted after normalizing each outcome relative to the lottery range. All certainty equivalents CE were linearly transformed to the $[0, 1]$ interval according

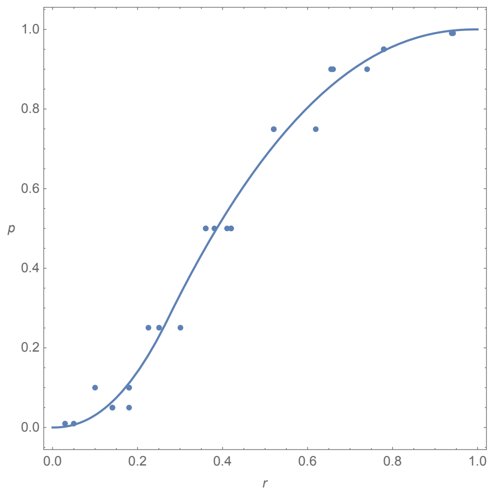

Figure 2: A single curve for all normalized ranges fitted with the CDF of the Two-Sided Power Distribution (Kotz and van Dorp, 2004). The horizontal axis represents lottery outcomes in each range normalized linearly into the interval $[0, 1]$. The vertical axis represents the probability of getting the x_u prize in each lottery.

to: $r = \frac{\text{CE}-x_l}{x_u-x_l}$; note that this transformation is in agreement with the Parducci's range principle. A single nonlinear function was then fitted with the (r, p) data pairs. The function was restricted to pass through the points $(0, 0)$ and $(1, 1)$. As demonstrated in Figure 2 a single utility function captures a lot of variation in the data. We will denote this function by D and refer to as the decision utility function (see Kontek, 2011).

[3]Tversky and Kahneman (1992) provide median Certainty Equivalents for a group of subjects only (individual data are not available). Technically these utilities are therefore utilities of the hypothetical median subject.

4 Axiomatic representation

4.1 Preliminaries

It is assumed that the decision maker cares only about probability distributions over outcomes. Thus we will exchangeably use lottery payoffs (random variables) and lotteries (probability distributions) as objects of choice, bearing in mind that both representations are equivalent given the assumed preference structure. In particular, lotteries are used in the axiomatic model in Section 4, because it is standard and natural in this context. Monotonicity and continuity, on the other hand, are analyzed in Section 5 in terms of lottery payoffs due to mathematical convenience.

Let $(S, \mathcal{S}, \Pi)$ be a probability space. Let $X = \mathbb{R}$ be the set of monetary prizes. A *lottery payoff* $\mathbf{x} : S \to X$ is a real-valued simple random variable, i.e. the image $\mathbf{x}(S)$ is a finite subset of $\mathbb{R}$. The set of all lottery payoffs is denoted by Lp. A lottery payoff, for which $\mathbf{x}(s) = x$, for all $s \in S$, for some $x \in X$ is called degenerate and is simply denoted by x.

A *lottery* is the distribution of a lottery payoff $\mathbf{x} \in Lp$ and is given by a function $P : X \to [0, 1]$, such that $P(x) = \Pi\left(\mathbf{x}^{-1}(x)\right)$, for $x \in \mathbf{x}(S)$. The set of all lotteries is the set of all probability distributions with finite support and is denoted by L. A *degenerate lottery* P^x is a lottery with one-element support, i.e. $P(x) = 1$ for some $x \in X$. The set of all degenerate lotteries is denoted by $L^d \subset L$. A lottery with more than one element in the support is called nondegenerate.

There is a preference relation over lotteries $\succsim \subset L \times L$. It is assumed that the decision maker cares only about probability distributions, i.e. two lottery payoffs with equal distribution are equivalent.[4] Having in mind this equivalence, a typical lottery (or lottery payoff) will be denoted by $(x_1, p_1; ...; x_n, p_n)$, where $n \in \mathbb{N}$ $x_i \in X$, $p_i \geq 0$, for $i \in \{1, 2, ..., n\}$, $\sum_{i=1}^n p_i = 1$.

Given a preference relation $\succsim \subset L \times L$ and a lottery P, the *Certainty Equivalent* of P is a number $\mathrm{CE}(P) \in X$ that satisfies: $P^{\mathrm{CE}(P)} \sim P$ (the conditions needed to ensure the existence of such number are specified in the next section). In particular, the Certainty Equivalent of a degenerate lottery P^x is equal to x. A mixing operation is performed on L, defined for every $P, Q \in L$ and every $\alpha \in [0, 1]$ as follows: $\alpha P + (1 - \alpha)Q \in L$ is given by: $(\alpha P + (1 - \alpha)Q)(x) = \alpha P(x) + (1 - \alpha)Q(x)$, $\forall x \in X$. For any lottery $P \in L$ define its *range* to be a convex hull of the support of P, i.e. $\mathrm{Rng}(P) = [\min(\mathrm{supp}(P)), \max(\mathrm{supp}(P))]$. The set of all lotteries with range equal to $[x_l, x_u] \subset X$, $x_l < x_u$ is given by: $L_{[x_l,x_u]} = \{P \in L : \mathrm{Rng}(P) = [x_l, x_u] \subset X\}$. The set of all degenerate lotteries with the support in $[x_l, x_u] \subset X$, $x_l < x_u$ is given by: $L^d_{[x_l,x_u]} = \{P^x \in L^d : x \in [x_l, x_u] \subset X\}$. We define the set of lotteries *comparable* within the $[x_l, x_u]$ range as $L^c_{[x_l,x_u]} =: L_{[x_l,x_u]} \cup L^d_{[x_l,x_u]}$, for $x_l < x_u$, $x_l, x_u \in X$. It consists of nondegenerate lotteries having range $[x_l, x_u]$ as well as degenerate lotteries whose support is in $[x_l, x_u]$.

Consider a nondegenerate lottery payoff $\mathbf{x}$ with range equal to $[x_l, x_u]$. Suppose that lottery payoff $\mathbf{y}$ has range equal to $[x_l, x'_u]$, where $x'_u > x_u$ and lottery payoff $\mathbf{z}$ has range equal to $[x'_l, x_u]$, where $x'_l < x_l$. We will say that $\mathbf{y}$ is an *upward range change* relative to $\mathbf{x}$ and $\mathbf{z}$ is a *downward range change* relative to $\mathbf{x}$. A lottery payoff $\mathbf{x}$ distributed according to the CDF F_x dominates lottery payoff $\mathbf{y}$ distributed according to the CDF F_y wrt FOSD, written $\mathbf{x} \succ_{\mathrm{FOSD}} \mathbf{y}$, if $F_x(t) \leq F_y(t)$ for all $t \in \mathbb{R}$ with strict inequality at least for some $t \in \mathbb{R}$. A sequence of lottery payoffs $\{\mathbf{x}_n\}_{n \in \mathbb{N}}$, such that each

[4] Formally, a preference relation over lottery payoffs $\succsim_S \subset Lp \times Lp$ is defined by: $\mathbf{x} \succsim_S \mathbf{y} \iff P \succsim Q$, where P, Q are the probability distributions of $\mathbf{x}, \mathbf{y}$, respectively.

$\mathbf{x}_n$ is distributed according to the CDF F_n, converges in distribution to the lottery payoff $\mathbf{y}$, distributed according to the CDF G (written $\mathbf{x}_n \xrightarrow{D} \mathbf{y}$), if the following holds: $\lim_{n\to\infty} F_n(x) = G(x)$, for every number $x \in \mathbb{R}$ at which G is continuous. The Certainty Equivalent functional $\mathrm{CE} : Lp \to X$ satisfies $\mathrm{CE}(\mathbf{x}) = \mathrm{CE}(P)$, where P is a lottery and $\mathbf{x}$ is the associated lottery payoff. The CE functional is *monotonic wrt FOSD* if $\mathbf{x} \succ_{\mathrm{FOSD}} \mathbf{y}$ implies $\mathrm{CE}(\mathbf{x}) > \mathrm{CE}(\mathbf{y})$. The CE functional is *continuous* if for every sequence of lottery payoffs $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, where each $\mathbf{x}_n$ is distributed according to F_n, converging in distribution to the lottery payoff $\mathbf{y}$ distributed according to G, the following holds: $\lim_{n\to\infty} \mathrm{CE}(\mathbf{x}_n) = \mathrm{CE}(\mathbf{y})$.

4.2 The range-dependent utility model

The range-dependent utility (RngDU) model generalizes the Expected Utility model by weakening its Continuity and Independence axioms. Instead of obtaining a single utility function u used to represent preferences between any pair of lotteries, different utility functions $u_{[x_l,x_u]}$ are allowed; each function represents preferences over lotteries having the same range. Expected Utility holds separately for lotteries with the same range. Preferences between lotteries with different ranges are represented by their Certainty Equivalent. The range-dependent utility model is not meant for prediction due to its high level of flexibility, it should rather be regarded a general conceptual framework.

A range-dependent preference relation $\succsim \subset L \times L$ satisfies the following axioms:

Axiom 1 (Weak Order). $\succsim$ *is complete and transitive.*

Axiom 2 (Within-Range Continuity). *Let $[x_l, x_u] \subset X$ be any interval such that $x_l < x_u$. For every $P, Q, R \in L^c_{[x_l,x_u]}$ it holds: $P \succ Q \succ R \Longrightarrow \exists \alpha, \beta \in (0, 1) : \alpha P + (1-\alpha)R \succ Q \succ \beta P + (1-\beta)R$.*

Axiom 3 (Within-Range Independence). *Let $[x_l, x_u] \subset X$ be any interval such that $x_l < x_u$. For every $P, Q, R \in L$ and $\alpha \in (0, 1)$ such that P, Q, $\alpha P + (1-\alpha)R, \alpha Q + (1-\alpha)R \in L^c_{[x_l,x_u]}$ it holds: $P \succsim Q \iff \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R$, $\forall \alpha \in [0, 1]$.*

Axiom 4 (Restricted Monotonicity). *For any $x, y \in X$ it holds: $x > y \iff P^x \succ P^y$.*

Axiom 4 states that more is better. Axioms 2 and 3 are the relaxed versions of Continuity and Independence of the EU model. Consider the following two choice problems, in which outcomes are measured in dollars:

$$(0, \tfrac{1}{2}; 1, \tfrac{1}{2}) \text{ vs. } (-1\mathrm{M}, \epsilon; 1, 1-\epsilon)$$
$$(0, \tfrac{1}{2}; 1, \tfrac{1}{2}) \text{ vs. } (0, 1-\epsilon; 1\mathrm{M}, \epsilon)$$

Suppose that someone chooses the lottery on the left in the first problem and the lottery on the right in the second problem even if ϵ is infinitesimally small. In both cases such behavior is consistent with Axiom 2 (it is vacuously true), but inconsistent with the Continuity axiom of the EU model. Now consider the following two choice problems where outcomes are measured in dollars:

$$(100, 0.5; 200, 0.5) \text{ vs. } (0, 0.1; 200, 0.9)$$
$$(0, 0.1; 100, 0.45; 200, 0.45) \text{ vs. } (0, 0.19; 200, 0.81)$$

Note that lotteries in the second problem are the reduced versions of the compound lotteries in which one obtains 0 with probability 0.1 and plays the corresponding lottery from the first problem with probability 0.9. Suppose that someone chooses the lottery on the left in the first problem and the lottery on the right in the second problem. Such behavior is consistent with Axiom 3 (it is vacuously true), but inconsistent with the Independence axiom of the EU model. We now state the theorem.

Theorem 1 (Range-dependent utility). *A preference relation $\succsim \subset L \times L$ satisfies axioms A1–A4 if and only if for every interval $[x_l, x_u] \subset X, x_l < x_u$ there exists a unique strictly increasing and surjective function $u_{[x_l,x_u]} : [x_l, x_u] \to [0, 1]$, such that for every pair of lotteries $P, Q \in L$ the following holds:*

$$P \succsim Q \iff \mathrm{CE}(P) \geq \mathrm{CE}(Q), \tag{1}$$

where the certainty equivalent is defined as:

a) $\mathrm{CE}(R) = u^{-1}_{\mathrm{Rng}(R)} \left[\sum_{x \in \mathrm{supp}(R)} R(x) u_{\mathrm{Rng}(R)}(x)\right]$*, for any $R \in L \setminus L^d$,*

b) $\mathrm{CE}(R) = x$*, for any $R \in L^d$, such that $R = P^x$.*

Proof. See Appendix 2. □

The range-dependent utility model may be regarded as an Expected Utility model with narrow framing, in which the decision-maker evaluates a given lottery relative to its range. It generalizes the standard Expected Utility model in the two following ways:

a. *The case of a universal range*: In real life there always exists a tiny chance to suddenly die or to find a billion dollars on the street. One can thus argue that the range of lotteries under consideration is always the same. If a person perceives the lottery range broadly (i.e. including those extreme events) there is only a single range-dependent utility function to represent choices between any pair of lotteries. Thus the decision maker behaves in accordance with the standard EU model[5]. On the other hand if a person perceives a lottery range narrowly (i.e. excluding those extreme events) there might be different utility functions applied to evaluate different lotteries. An important implication would be that such people might violate the standard EU model if lotteries under consideration have different ranges.

b. *The case of consequentialism*: Axioms 1-4 guarantee that the Expected Utility of wealth model is a special case of the range-dependent utility model. Here we explicitly show how to construct the equivalent range-dependent utility model for any Expected Utility of wealth model. Let $\succsim^{\mathrm{w}}$ be a preference relation defined over lotteries in which the prizes are interpreted as the final wealth levels; such lotteries award prize $W + x$ with probability $P(x)$, for $P \in L$ and initial wealth $W \in \mathbb{R}$. Let $\succsim^{\mathrm{w}}$ be represented by a strictly increasing vNM utility function u. For each interval $[x_l, x_u] \subset X$, $x_l < x_u$, define the range-dependent utility function $u_{[x_l,x_u]} : [x_l, x_u] \to [0, 1]$,

[5]Formally, assume that $X \in \mathbb{R}$ is compact so that there exist $x_w, x_b \in \mathbb{R}$ such that for any $P \in L$, $P^{x_b} \succsim P \succsim P^{x_w}$. In the standard EU model each lottery is evaluated relative to P^{x_b} and P^{x_w}. Consider now the range-dependent utility decision maker who always allows tiny chance of x_w and x_b. Let P be the lottery with range $[x_l, x_u] \subset X$, $x_l < x_u$. Let $P_{\text{ext}} := 0.5P^{x_b} + 0.5P^{x_w}$. Then instead of lottery P, the decision maker can be modeled as evaluating the following lottery: $P_\epsilon := (1 - \epsilon)P + \epsilon P_{\text{ext}}$, where $\epsilon > 0$. The range of such lottery is $[x_w, x_b]$, for every $\epsilon > 0$, but since X is bounded P_ϵ converges to P as ϵ tends to zero. Thus the range-dependent utility decision maker who evaluates P_ϵ instead of P behaves in the limit in the same way as the EU decision maker who evaluates P.

such that: $u_{[x_l,x_u]}(x) = \frac{u(W+x)-u(W+x_l)}{u(W+x_u)-u(W+x_l)}$. The model defined this way is equivalent to the EU of wealth model by construction.

Range is a crucial element of the range-dependent utility model. In our setup we assume that it comes with the description of the lottery. In real life it is often not the case. When a lottery is experienced through repeated sampling its range depends on the specific experience and the decision maker's beliefs. For instance, many people realize that unfavorable events are possible just after being burgled or suffering loss or damage, and, consequently, decide to buy an insurance even though they were previously reluctant to do so. The more general approach would be to analyze range-dependence in the context of uncertainty with range being the interval between the lowest and the highest consequence in the subjective lottery support. Such an approach may help explaining the description-experience gap.

4.3 The decision utility model

The decision utility model is a special case of the range-dependent utility framework which is used for prediction. Apart from the four range-dependent utility axioms, it requires the additional axiom, namely Scale and Shift invariance. It allows to have a single decision utility function defined on the interval $[0, 1]$ (i.e. the normalized lottery outcomes) instead of a great multiplicity of the range-dependent utility functions (one for each range). The psychological motivation for the Shift and Scale Invariance is given by the Weber Law, which states that increasing the stimuli range increases proportionaly the Just Noticable Difference.

Definition 1. *For a lottery $P \in L$ define its α, β-transformation $P_{\alpha,\beta} \in L$, such that $P(x) = P_{\alpha,\beta}(\alpha x + \beta)$, for all $x \in X$, where $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$, .*

Axiom 5 (Scale and Shift Invariance). *Let $[x_l, x_u] \subset X$ be any interval such that $x_l < x_u$. For any two lotteries $P, Q \in L^c_{[x_l,x_u]}$ and any $\alpha > 0, \beta \in \mathbb{R}$ such that $P_{\alpha,\beta}, Q_{\alpha,\beta} \in L$ it holds: $P \succsim Q \iff P_{\alpha,\beta} \succsim Q_{\alpha,\beta}$.*

Theorem 2 (Decision utility)**.** *A preference relation $\succsim \subset L \times L$ satisfies axioms A1–A5 if and only if there exists a unique strictly increasing and surjective function $D : [0, 1] \to [0, 1]$, such that for every pair of lotteries $P, Q \in L$ the following holds:*

$$P \succsim Q \iff \mathrm{CE}(P) \geq \mathrm{CE}(Q), \tag{2}$$

where the certainty equivalent is defined as:

a) $\mathrm{CE}(R) = x_l + (x_u - x_l)D^{-1}\left[\sum_{x\in \mathrm{supp}(R)} R(x)D\left(\frac{x-x_l}{x_u-x_l}\right)\right]$*, for any $R \in L \setminus L^d$, where* $x_l = \min(\mathrm{Rng}(R))$*,* $x_u = \max(\mathrm{Rng}(R))$*,*

b) $\mathrm{CE}(R) = x$*, for any $R \in L^d$, such that $R = P^x$.*

Proof. In Appendix 2. □

Note that for any nondegenerate lottery R one can find a binary lottery $(x_l, 1 - p'; x_u, p')$ that is equivalent (preference-wise) to R. The probability p', referred to as the equivalent probability of R, is given by the expression in the squared brackets in the $\mathrm{CE}(R)$ definition in point a) both in Theorem

1 and 2. When R is a binary lottery of the form $(x_l, 1-p; x_u, p)$, this expression reduces to p: since $D(0) = 0$, and $D(1) = 1$, we have $(1-p)D\left(\frac{x_l - x_l}{x_u - x_l}\right) + pD\left(\frac{x_u - x_l}{x_u - x_l}\right) = (1-p)0 + p1 = p$.

The axiom of Shift and Scale Invariance is crucial in the decision utility model. This axiom, added to axioms 1–4, implies that the family of range-dependent utility functions $\left(u_{[x_l,x_u]}\right)_{x_l,x_u \in \mathbb{R}:\ x_l < x_u}$ is induced from a single decision utility function D by taking:

$$u_{[x_l,x_u]}(x) := D\left(\frac{x - x_l}{x_u - x_l}\right),\ \forall x \in [x_l, x_u].$$

This makes the model operational — there is only one function instead of many; eliciting this function is sufficient to predict choices involving different lottery ranges. The model gives strong testable predictions as it has only one free element to choose, i.e. the shape of D.

Some may argue that this model is not particularly good at capturing the evidence where lottery ranges vary a lot. For instance, somebody may well be indifferent between a certain \$40 and a 50% chance of winning \$100, but will definitely prefer a certain \$40 million to a 50% chance of winning \$100 million. Therefore the axiom of Shift and Scale Invariance should be treated as benchmark case.[6]

5 Continuity and monotonicity properties in the decision utility model

Consider an interval $[x_l, x_u] \subset \mathbf{X}$, where $x_l < x_u$. The range-dependent utility model confined to the set of lotteries comparable within this interval, i.e. in the set $L^c_{[x_l,x_u]}$, is equivalent to the Expected Utility model with a strictly increasing utility function u if and only if the following holds: $u_{[x_l,x_u]}(x) = u(x)$, for all $x \in [x_l, x_u]$. This is true separately for each range $[x_l, x_u] \subset X$. Hence the range-dependent utility model is consistent with the First-Order Stochastic Domination for lotteries comparable within the same range. However, monotonicity violations may still occur for nondegenerate lotteries having different ranges.

Due to the axiom of Scale and Shift Invariance, the shape of a given utility function for one lottery range uniquely determines the shapes of the utility functions for all other ranges. This restriction is, however, not sufficient to exclude monotonicity violations. Consider the following two lottery payoffs $\mathbf{x}^{\mathrm{do}} = (0, \epsilon; 10, \frac{1}{2} - \epsilon; 20, \frac{1}{2})$, $\mathbf{y} = (10, \frac{1}{2}; 20, \frac{1}{2})$, where ϵ is a very small but positive probability. Note that $\mathbf{x}^{\mathrm{do}}$ should have a lower CE value because it is dominated by $\mathbf{y}$ wrt FOSD. However, assuming the decision utility function of the form $D(x) = \sqrt{x}$, for $x \in [0, 1]$, $\mathrm{CE}(\mathbf{x}^{\mathrm{do}})$ approaches 14.57 as ϵ tends to zero whereas $\mathrm{CE}(\mathbf{y}) = 12.5$; hence monotonicity is violated.

Intuitively the problem may be explained as follows. When ϵ tends to zero, all the probability mass is concentrated on the outcomes of 10 and 20 in both $\mathbf{x}^{\mathrm{do}}$ and $\mathbf{y}$. However as these lottery payoffs have different ranges, the probability mass is spanned over the intervals $\left[\frac{1}{2}, 1\right]$ and $[0, 1]$ of their respective

[6]Observe that in the decision utility model of Theorem 2 the shape of the utility function remains the same for different ranges. On the other hand, in the range-dependent utility model of Theorem 1 the shape of the utility function may change even due to small changes in the lottery range. It is possible to define an intermediate model, in which the shape of the utility function is allowed to change considerably only for big changes in the lottery range. Such a model may be defined as follows: The family of range-dependent utility functions $\left(u_{[x_l,x_u]}\right)_{x_l,x_u \in \mathbb{R}:\ x_l < x_u}$ is induced from a single decision utility function $D : [0, 1] \to [0, 1]$ (that captures range effects) and a single utility function $v : X \to [0, 1]$ (which captures attitudes towards lottery consequences) by taking: $u_{[x_l,x_u]}(x) = D\left(\frac{v(x) - v(x_l)}{v(x_u) - v(x_l)}\right)$. If v belongs to CRRA class, the resulting model exhibits Scale Invariance (but not necessarily Shift Invariance). Similarly, if v belongs to CARA class, the resulting model exhibits Shift Invariance (but not necessarily Scale Invariance). Such a model could explain the behavior described in the text.

normalized ranges. A higher overall level of risk aversion in the interval $\left[\frac{1}{2}, 1\right]$ than in $[0, 1]$ is thus required to ensure that $\mathrm{CE}(\mathbf{x}^{\mathrm{do}}) < \mathrm{CE}(\mathbf{y})$. This condition fails for the decision utility function of the form $D(x) = \sqrt{x}$, which belongs to the class of decreasing absolute risk aversion.

A similar example of monotonicity violation may be presented for the following two lottery payoffs: $\mathbf{x}^{\mathrm{up}} = (10, \frac{1}{2}; 20, \frac{1}{2} - \epsilon; 30, \epsilon)$, $\mathbf{y} = (10, \frac{1}{2}; 20, \frac{1}{2})$ where $\frac{1}{2} > \epsilon > 0$ and the decision utility function of the form $D(x) = 1 - \sqrt{1 - x}$, for $x \in [0, 1]$. Lottery payoffs $\mathbf{x}^{\mathrm{do}}$ and $\mathbf{x}^{\mathrm{up}}$ are a downward and an upward range change relative to $\mathbf{y}$ (See Section 4.1 for a formal definition). The examples above show that monotonicity violations need to be excluded for both downward and upward range changes relative to any lottery.

5.1 Necessary and sufficient conditions

The main result of this section is the identification of necessary and sufficient conditions that guarantee monotonicity and continuity in the decision utility model. As shown below monotonicity is satisfied for a broad class of decision utility functions, whereas continuity is violated in general.

Definition 2. *Given the decision utility function D, define the backward decision utility function $C : [0, 1] \to [0, 1]$ such that $C(x) = 1 - D(1 - x)$, for all $x \in [0, 1]$.*

It is easy to verify that the decision utility model in Theorem 2 point a) can be expressed equivalently in terms of D (the original formulation) as well as in terms of C:

$$\mathrm{CE}(R) = x_l + (x_u - x_l)D^{-1}\left[\sum_{x \in \mathrm{supp}(R)} R(x) D\left(\frac{x - x_l}{x_u - x_l}\right)\right],$$
$$\mathrm{CE}(R) = x_u - (x_u - x_l)C^{-1}\left[\sum_{x \in \mathrm{supp}(R)} R(x) C\left(\frac{x_u - x}{x_u - x_l}\right)\right]. \tag{3}$$

Function C in (3) evaluates a given relative lottery outcome backwards in comparison to function D. Based on the two functions, D and C, we define two local measures of relative risk attitudes and two benchmark decision utility functions.

Definition 3. *Assume that the decision utility function D is twice differentiable on the interval $[0, 1]$. (the same is automatically true for function C. We define the Relative Risk Aversion for D and for C, denoted $\mathrm{RRA_D}, \mathrm{RRA_C} : [0, 1] \to (-\infty, 1)$ as:*

$$\mathrm{RRA_D}(x) = -\frac{xD''(x)}{D'(x)}, \quad x \in [0, 1],$$
$$\mathrm{RRA_C}(x) = -\frac{xC''(x)}{C'(x)}, \quad x \in [0, 1].$$

Lemma 1. *The following two equivalences hold:*

a) $\mathrm{RRA_D}$ is a constant function if and only if $D(x) = x^\alpha$, for $x \in [0, 1]$, where $\alpha > 0$,

b) $\mathrm{RRA_C}$ is a constant function if and only if $D(x) = 1 - (1 - x)^\beta$, for $x \in [0, 1]$, where $\beta > 0$.

Proof. See Appendix 2. □

Remark: As the decision utility function is bounded on the interval $[0,1]$, the conditions $\alpha > 0$ and $\beta > 0$ in Lemma 1 imply that the $\mathrm{RRA_D}$ and $\mathrm{RRA_C}$ values are restricted to be less than 1.

We now state the main result of this section concerning monotonicity and continuity:

Theorem 3 (Monotonicity and continuity in the decision utility model)**.** *In the decision utility model, monotonicity (continuity) holds if and only if the decision utility function simultaneously satisfies the following two conditions:*

a) $\mathrm{RRA_D}(x)$ *is non-decreasing (constant) for all* $x \in [0,1]$

b) $\mathrm{RRA_C}(x)$ *is non-decreasing (constant) for all* $x \in [0,1]$

Proof. See Appendix 2. □

Condition a) in the above theorem prevents from monotonicity (continuity) violations involving upward range changes, whereas condition b) prevents from violations involving downward range changes. Monotonicity is relatively easy to obtain as the conditions have to be satisfied with inequalities ($\mathrm{RRA_D}$ and $\mathrm{RRA_C}$ non-decreasing). As a consequence, a wide variation in the shape of the decision utility function is allowed. Continuity is much harder to obtain as the relevant conditions have to be satisfied with equality ($\mathrm{RRA_D}$ and $\mathrm{RRA_C}$ constant). As a consequence, the decision utility model is discontinuous in general:

Corollary 1. *In the decision utility model, continuity holds if and only if* $D(x) = x$*, for all* $x \in [0,1]$.

Proof. Obvious: the only function D which is both power and inverse power is linear. □

A similar conclusion holds for the general model of range-dependence in Theorem 1. The only nontrivial range-dependent utility model satisfying continuity coincides with the standard Expected Utility model.

Due to the discontinuity feature, Scale and Shift Invariance assumed in the decision utility model does not imply risk neutrality. This way the model nontrivially exhibits Constant Risk Aversion of Safra and Segal (1998). This is not the case in the standard Expected Utility model: scale-invariance is equivalent to CARA and shift-invariance to CRRA utility, so that both simultaneously boil down to risk neutrality (Pratt, 1964, Lewandowski, 2013).

5.2 Examples of decision utility functions satisfying monotonicity conditions

In what follows we analyze monotonicity conditions algebraically for two parametric families of functions: the CDF of the Beta distribution and the CDF of the Two-Sided Power Distribution.

Example 1. *The decision utility function takes the form of the CDF of the Beta distribution:*

$$D(x) = A \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt, \quad \text{for } x \in [0,1]. \tag{4}$$

where $A = \frac{1}{\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt}$*, and* $\alpha, \beta > 0$*. The Relative Risk Aversion functions for* D *and corresponding* C *are as follows:*

$$\begin{aligned} \mathrm{RRA_D}(x) &= -(\alpha-1) + (\beta-1)\tfrac{x}{1-x}, \quad && \text{nondecreasing iff } \beta \geq 1 \\ \mathrm{RRA_C}(x) &= (\alpha-1)\tfrac{x}{1-x} - (\beta-1), \quad && \text{nondecreasing iff } \alpha \geq 1 \end{aligned}$$

Monotonicity conditions are satisfied in four special cases:

a) $\alpha = \beta = 1$ *corresponds to* $D(x) = x$ *(linear),*

b) $\alpha > 1, \beta = 1$ *corresponds to* $D(x) = x^{\alpha}, \alpha > 1$ *(convex power function),*

c) $\alpha = 1, \beta > 1$ *corresponds to* $D(x) = 1 - (1 - x)^{\beta}, \beta > 1$ *(concave inverse power function),*

d) $\alpha, \beta > 1$ *corresponds to all S-shaped functions in this family.*

All functions for which $\alpha \in (0, 1)$ *or* $\beta \in (0, 1)$ *are excluded. In particular, all inverse S-shaped functions, where* $\alpha, \beta \in (0, 1)$*, do not satisfy the monotonicity conditions.*

Example 2. *The decision utility function takes the form of the CDF of the Two-Sided Power Distribution (Kotz and van Dorp, 2004):*

$$D(x) = \begin{cases} x_0 \left(\frac{x}{x_0}\right)^{\alpha}, & 0 \leq x \leq x_0, \\ 1 - (1 - x_0)\left(\frac{1-x}{1-x_0}\right)^{\alpha}, & x_0 \leq x \leq 1, \end{cases} \tag{5}$$

where $x_0 \in (0, 1)$*,* $\alpha > 0$*. The Relative Risk Aversion functions for* D *and corresponding* C *are as follows:*[7]

$$\mathrm{RRA_D}(x) = \begin{cases} 1 - \alpha, & 0 \leq x < x_0 \\ \frac{x}{1-x}(\alpha - 1), & x_0 < x \leq 1 \end{cases}, \qquad \text{nondecreasing iff } \alpha \geq 1$$

$$\mathrm{RRA_C}(x) = \begin{cases} 1 - \alpha, & 0 \leq x < 1 - x_0 \\ -\frac{x}{1-x}(1 - \alpha), & 1 - x_0 < x \leq 1 \end{cases}, \qquad \text{nondecreasing iff } \alpha \geq 1$$

Monotonicity conditions are satisfied in four special cases:

a) $\alpha = 1$ *corresponds to* $D(x) = x$ *(linear),*

b) $x_0 = 1, \alpha > 1$ *corresponds to* $D(x) = x^{\alpha}, \alpha > 1$ *(convex power function),*

c) $x_0 = 0, \alpha > 1$ *corresponds to* $D(x) = 1 - (1 - x)^{\alpha}, \alpha > 1$ *(concave inverse power function),*

d) $x_0 \in (0, 1), \alpha > 1$ *corresponds to all S-shaped functions in this class.*

All inverse S-shaped functions in this class, for which $\alpha \in (0, 1)$*, are excluded.*

It is interesting that the same conclusions (i.e. convex power, concave inverse power, and all S-shaped functions within each of the classes) result from analyzing the two different classes of functions.

5.3 Monotonicity and continuity in the Marschak-Machina triagle

In this section we analyze monotonicity and continuity graphically in the Marschak-Machina triangle. Consider 3-outcome lotteries $(x_1, p_1; x_2, p_2; x_3, p_3)$, where $x_1 < x_2 < x_3$, $p_i \in [0, 1]$ for $i \in \{1, 2, 3\}$ and $\sum_{i=1}^{3} p_i = 1$. Figure 3 shows the Marschak-Machina triangles with the indifference curves that correspond to different forms of the decision utility function. In all cases the indifference curves inside

[7] Note that the functions $\mathrm{RRA_D}$ and $\mathrm{RRA_C}$ are not defined for $x = x_0$, because the second derivative $D''(x_0)$ does not exists, i.e. the left derivative is different than the right derivate: $D''_{-}(x_0) \neq D''_{+}(x_0)$. Fortunately, Theorem 3 can be modified in this case to require $D''_{-}(x_0) < D''_{+}(x_0)$ as well as $C''_{-}(1 - x_0) < C''_{+}(1 - x_0)$.

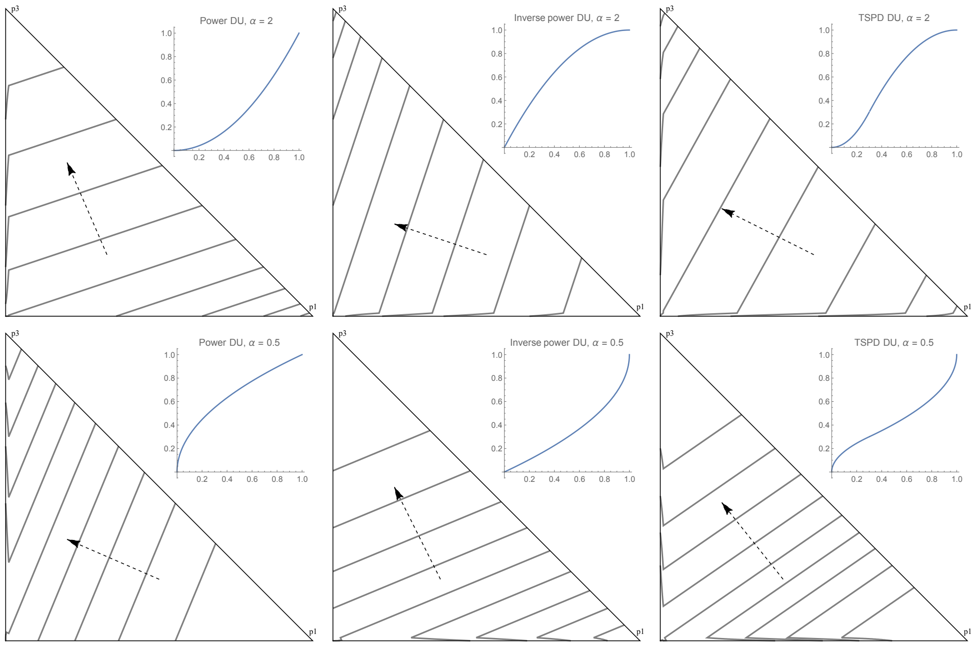

Figure 3: Indifference curves in the Marschak-Machina triangles for different decision utility functions. In each of the triangles, the horizontal axis measures the probability of the worst and the vertical axis – of the best lottery outcome. Lotteries inside the triangles and on the hypotenuse have the same range and hence indifference curves are straight parallel lines as implied by Expected Utility Theory. Moving from the inside of the triangle to its legs changes the lottery range, which leads to discontinuous jumps in the indifference curves. For power and inverse power functions, jumps occur at only one of the legs. Monotonicity is satisfied for the three functions depicted in the upper raw and violated for those three depicted in the lower raw. Note! Jumps are presented as sharp kinks rather than real discontinuities for better visibility.

the triangle are straight parallel lines as implied by Expected Utility Theory. Note that the range is $[x_1, x_3]$ for lotteries located inside the triangle and on the hypotenuse, $[x_2, x_3]$ for lotteries located on the vertical leg, and $[x_1, x_2]$ for lotteries located on the horizontal leg. Therefore, the indifference curves are generally discontinuous at both legs of the triangle but not on the hypotenuse (two rightmost panels in Figure 3). There are two limiting cases, for which discontinuity occurs at only one of the triangle legs:

a) the power function $D(x) = x^{\alpha}$, $\alpha > 0$ (two leftmost panels), where the indifference curves are discontinuous only at the vertical leg,

b) the inverse power function $D(x) = 1-(1-x)^{\beta}$, $\beta > 0$ (two middle panels), where the indifference curves are discontinuous only at the horizontal leg.

The features described above can be derived as a direct consequence of Theorem 3 and Lemma 1: the power function satisfies continuity with respect to upward range changes and violates continuity with respect to downward range changes. The opposite holds for the inverse power function: continuity is satisfied wrt downward and violated wrt upward range changes.

When moving from the inside of the triangle to the triangle legs, the jumps of indifference curves can either be directed towards the origin, in which case they satisfy monotonicity (three upper panels), or away from the origin (three lower panels), in which case they violate monotonicity. The main conclusion is that it is "easiest" to satisfy monotonicity if the decision utility function is S-shaped, i.e. it is characterized by low marginal utility at the edges of its domain (close to 0 and 1) and high marginal utility in the middle (upper right panel). Per contra monotonicity is violated by all inverse S-shaped decision utilities (lower right panel).

6 Decision utility vs. probability weighting

It has been argued in this paper that the typical decision utility function is S-shaped. It is characterized by low marginal utility at the edges of its domain and high marginal utility in the middle. Looking at the inverse of the decision utility function, the arguments of which are (equivalent) probabilities, one can note high sensitivity to probabilities close to 0 or 1 and low sensitivity for the middle probabilities. This feature resembles the probability weighting function of CPT. One could thus have an impression that the decision utility model is merely an algebraic sleight of hand with the (inverse) decision utility function in place of the probability weighting function. This impression is only partially true and only in the context of binary lotteries.

6.1 Observational equivalence for binary lotteries

There is no probability weighting in the decision utility model. By contrast (cumulative) probabilities are treated nonlinearly in rank-dependent utility models, such as Dual Theory of choice (Yaari, 1987) or CPT (Tversky and Kahneman, 1992).[8] It turns out, however, that in the case of binary lotteries the decision utility model and Dual Theory (or CPT with a linear value function) are observationally equivalent. To see this observe that for a binary lottery payoff $(x_l, 1-p; x_u, p)$, the decision utility model given by Theorem 2 and Dual Theory with a (de-) cumulative probability weighting function w can be written, respectively, as follows:

$$\begin{aligned} \mathrm{CE}(\mathbf{x}) &= x_l + (x_u - x_l)D^{-1}(p), \qquad (6) \\ \mathrm{CE}(\mathbf{x}) &= x_l + (x_u - x_l)w(p). \end{aligned}$$

Note that the two models give the same predictions if and only if $D^{-1}(p) = w(p)$ for every $p \in [0,1]$. The important implication of this fact is that the two models cannot be distinguished based on the evidence involving binary lotteries only. It also means that such evidence, usually brought forward to support the idea of probability weighting, supports the idea of decision utility to the same extent. The equivalence between decision utility and cumulative probability weighting no longer holds for multi-outcome lotteries. This allows for model discrimination.

6.2 Descriptive accuracy of the model

As discussed above, in the case of binary gambles the decision utility model is empirically equivalent to the Yaari model. The advantage of CPT over the decision utility model results from an additional function (value function v) in CPT. In the case of Tversky and Kahneman (1992) data adding this function reduces SSE by 10% (see Appendix 1 for details).

[8] For lotteries involving exclusively non-negative (or exclusively non-positive) outcomes CPT differs from Dual Theory only by having an additional value function for outcomes; when this function is linear both models are equivalent.

In the case of three-outcome lotteries, the important feature of the decision utility model illustrated in Figure 3 is that the indifference curves in the Marschak-Machina triangle are straight parallel lines inside the triangle as per the EU hypothesis and are discontinuous at the legs (but not at the hypotenuse). On the other hand, CPT predicts that the indifference curves are concave in the upper left and convex in the lower right part of the MM-triangle. Importantly, these indifference curves are nonlinear but smooth everywhere.

There is a vast experimental literature that supports predictions of the decision utility model. Harless (1992) finds that systematic violations of expected utility "disappear when lotteries are nudged inside the triangle". This suggests that the indifference curves inside the triangle are straight parallel lines. The same conclusion can be derived from other works. Conlisk (1989) analyzes three variants of the Allais paradox and shows that the EU violations are largely diminished when the lotteries involved are located inside the MM-triangle. He concludes that the results "favor the certainty effect over the fanning out hypothesis for at least the case of straight indifference lines". Sopher and Gigliotti (1993) observe that the patterns in the off-border treatment are significantly different than those in the on-border treatment. They find that in the former case expected utility performs well. Harless and Camerer (1994) note that which theories are best depends on whether lotteries being compared have the same support (EU fits better) or not (EU fits poorly). Cohen (1992) states that Camerer (1989) within-subject analysis confirms that most violations of EU involve lotteries at the legs of the MM-triangle and they are never significant in the interior of the triangle and on the hypotenuse. The claim that EU performs well inside the MM-triangle is also supported by Hey and Orme (1994).

In a recent experimental work Kontek (2016) estimates nonparametrically indifference curves using certainty equivalents. Many of the lotteries considered were located close to the MM-triangle boundaries. The main finding is that the indifference curves exhibit considerable jumps when moving from the inside of the triangle to its legs. These jumps are directed towards the origin. Econometrically, this effect is characterized by a sudden change in the slopes of the indifference curves. This result is consistent with the decision utility model satisfying monotonicity. Moreover, Kontek used the same data to test six decision-making models. The decision utility model that correctly predict jumps of indifference curves at the triangle legs offered the best estimation results. The CPT model, which predicts smooth and continuous indifference curves, was only ranked fourth with the Sum of Squared Errors of more than 50% greater than that for the decision utility model even though it uses one parameter more (the one corresponding to the value function). These results suggest that boundary effects at the legs of the triangle capture most variation in the data. Nonlinearity of the indifference curves in the interior of the triangle is only a second-order effect. The decision utility model performs well because it preserves the Expected Utility Theory features inside the triangle and, at the same time captures the boundary effects.

7 Accommodating EU paradoxes

The decision utility model explains some well known Expected Utility paradoxes without recourse to probability weighting. The paradoxes considered here are: the Common Ratio effect and the Allais paradox, the coexistence of insurance and gambling (including the powerball case), and the Rabin (2000) paradox.

7.1 The Allais paradox and the common ratio effect

For $y > x > 0$, $1 > p > q > 0$, $\frac{q}{p} > p$, the following two pairs of preference patterns are general versions of the two well-known paradoxes:

$$\begin{array}{ll} \text{the Allais paradox} & \left\{ \begin{array}{rll} (y,q;x,1-p;0,p-q) & \prec (x,1), & \text{(A)} \\ (y,q;0,1-q) & \succ (x,p;0,1-p). & (*) \end{array} \right. \\ \text{the Common Ratio effect} & \left\{ \begin{array}{rll} \left(y,\frac{q}{p};0,1-\frac{q}{p}\right) & \prec (x,1), & \text{(B)} \\ (y,q;0,1-q) & \succ (x,p;0,1-p). & (*) \end{array} \right. \end{array}$$

By stating the paradoxes jointly in the above way, we can draw interesting conclusions. Under EU the conditions (A) and (B) are equivalent but they are inconsistent with $(*)$:

$$\underbrace{\frac{u(W+x)}{u(W+y)} < \frac{q}{p}}_{\text{(A),(B)}} \overbrace{< \frac{u(W+x)}{u(W+y)}}^{(*)}$$

where $W \geq 0$, $u(W) = 0$ and u is the vNM utility function.

Under the decision utility model (A) and (B) are also equivalent. This time however they are consistent with $(*)$:

$$\left.\begin{array}{rll} yD^{-1}\left(q+(1-p)D\left(\frac{x}{y}\right)\right) < & x & \text{(A)} \\ yD^{-1}(q) > & xD^{-1}(p) & (*) \\ yD^{-1}\left(\frac{q}{p}\right) < & x & \text{(B)} \end{array}\right\} \iff \underbrace{D^{-1}\left(\frac{q}{p}\right) < \frac{x}{y}}_{\text{(A),(B)}} \overbrace{< \frac{D^{-1}(q)}{D^{-1}(p)}}^{(*)}$$

There is a single condition to predict both kinds of paradoxes; the function D should be flat enough in the upper part of its domain, i.e. in the interval $\left[D^{-1}\left(\frac{q}{p}\right), 1\right]$, and steep enough in the middle part of its domain, i.e. in the interval $\left[D^{-1}(q), D^{-1}(p)\right]$. These restrictions are compatible with an S-shaped decision utility function D. Consider for example $y = \$4000, x = \$3000, p = 0.25, q = 0.2$. In order to predict both paradoxes the decision utility model requires: $D^{-1}(0.8) < 0.75 < \frac{D^{-1}(0.2)}{D^{-1}(0.25)}$, which is satisfied by the decision utility function estimated in Appendix 1.

Dual Theory can predict both paradoxes as well, but, interestingly, the conditions (A) and (B) are not equivalent any more. This leads to separate conditions for the Allais and the CR paradoxes:

the Allais paradox:

$$\underbrace{\frac{w(q)}{w(q)+1-w(1-p+q)} < \frac{x}{y}}_{\text{(A)}} \overbrace{< \frac{w(q)}{w(p)}}^{(*)},$$

the Common Ratio effect:

$$\underbrace{w\left(\frac{q}{p}\right) < \frac{x}{y}}_{\text{(B)}} \overbrace{< \frac{w(q)}{w(p)}}^{(*)},$$

where w is the (de-) cumulative probability weighting function. Note that the condition for the CR effect is the same as in the decision utility model if one replaces w with D^{-1}. This is true because it concerns binary lotteries in which case both models are observationally equivalent. On the contrary, the conditions for the Allais paradox are different in the two models because one of the Allais lotteries involves three outcomes. The conditions for both effects differ also in the case of CPT, where x and y are replaced by $v(x)$ and $v(y)$, respectively, where v is the CPT value function.

7.2 The coexistence of gambling and insurance

Consider the fair gambling problem. There is a small chance $p > 0$ to win a large prize $P > 0$. The price for a lottery ticket is actuarily fair, i.e. equals pP. The decision maker decides whether to buy the lottery ticket or not. Similarly, consider the fair insurance problem. There is a small probability $p > 0$ that your property (valued at $H > 0$) is destroyed completely. The price for a full insurance of the property is actuarily fair, i.e. equals pH. The decision maker decides whether to insure the property or not. A coexistence of insurance and gambling arises when an individual chooses to gamble and insure at the same time. It may be represented by the following pattern of preferences:

$$(P - pP, p; -pP, 1-p) \succ (0, 1),$$
$$(H, 1-p; 0, p) \prec (H - pH, 1).$$

This pattern is predicted by the decision utility model if the following conditions are satisfied:

$$\begin{array}{rl} -pP + PD^{-1}(p) > & 0 \\ HD^{-1}(1-p) < & H - pH \end{array} \Rightarrow p > \max(D(p), 1 - D(1-p))$$

The conditions require that the decision utility function is flat at the boundaries of its domain, i.e. close to 0 and close to 1. It is the case for an S-shaped function, which is graphically illustrated in Figure 4.

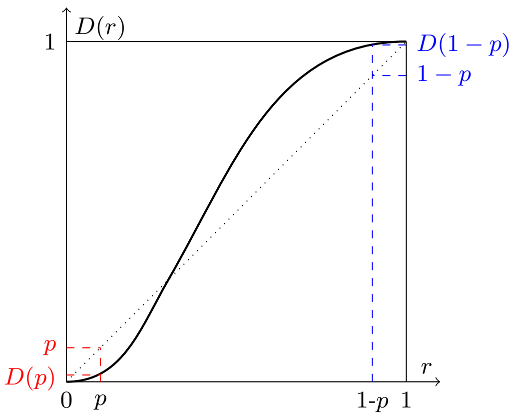

Figure 4: Coexistence of insurance and gambling is possible due to an S-shaped decision utility. The formulas in the south west apply to gambling – no gambling and the formulas in the north east to insurance – no insurance comparison.

Dual Theory also predicts the above preference patterns because lotteries considered are binary. Algebraically the difference is small – in the above conditions one needs to replace D^{-1} with w – but the underlying psychological interpretation (probability transformation vs. outcome transformation) differs completely. Note that the latter explanation is close to that of Markowitz (1952) who uses an S-shaped utility function (separate for gains and losses) to predict the paradox.

Powerball lottery case

Closely related to the coexistence of insurance and gambling are the examples considered in the Introduction. In what follows we model them in a highly simplified way. Consider two Powerball lotteries: one which offers a huge prize P with a tiny probability p, or otherwise nothing; and another which offers an even larger prize λP with an even tinier probability $\frac{p}{\lambda k}$, where $\lambda, k > 1$. Note that the expected value of the second lottery is lower although its range is larger. Then, consider two projects. In the first you may either suffer a big loss L with a tiny probability p or keep the *status quo* otherwise. In the second you may suffer an even bigger loss (say of a bankrupting severity) λL with an even tinier probability $\frac{p}{\lambda k}$, or keep the *status quo* otherwise, where $\lambda, k > 1$. Note that the range of the second project as well as its expected value are larger than in the first project. In analogy to the example given in the introduction, we expect that many people will prefer the lottery with a higher jackpot value and at the same time will avoid the project with a higher loss, even though their expected values are lower than those of the respective alternatives. This may be expressed as:

$$(P, p; 0, 1-p) \prec \left(\lambda P, \tfrac{p}{\lambda k}; 0, 1 - \tfrac{p}{\lambda k}\right)$$
$$(-L, p; 0, 1-p) \succ \left(-\lambda L, \tfrac{p}{\lambda k}; 0, 1 - \tfrac{p}{\lambda k}\right)$$

The decision utility model predicts this pattern of preferences under the following conditions:

$$D^{-1}(p) < \lambda D^{-1}\left(\tfrac{p}{\lambda k}\right)$$
$$C^{-1}(p) < \lambda C^{-1}\left(\tfrac{p}{\lambda k}\right)$$

Assuming that D is of the form of the Two-Sided Power Distribution (with parameters x_0 and $\alpha > 0$) considered in Example 2 and $p < x_0 < 1-p$, it must be that $\alpha > \frac{\log(\lambda k)}{\log(\lambda)}$ in order to satisfy the conditions stated above. This in turn implies that D is "sufficiently" S-shaped – it is sufficiently convex for $[0, x_0]$ and sufficiently concave for $[x_0, 1]$.

Can the standard EU model explain this preference pattern as well? Yes, but this would require that the decision maker is sufficiently risk loving for wealth levels above his current one, i.e. over the interval $[W, W + \lambda P]$, and sufficiently risk averse for wealth levels below his current one, i.e. over the interval $[W - \lambda L, W]$. It is well accepted that such explanation based on an inverse-S shaped utility of wealth is far-fetched (see also Friedman and Savage, 1948 and its critical evaluation by Markowitz, 1952).

7.3 The Rabin paradox

Rabin (2000) has found that the standard Expected Utility model implies the following behavior: if an individual rejects an equal chance gamble of winning \$110 or losing \$100 at all wealth levels below \$300 000 (which Rabin finds plausible), then the same individual must also reject an equal chance gamble of losing \$1 000 or winning an arbitrarily high sum of money at any wealth level (which Rabin finds implausible). Let W denote initial wealth of the decision maker. In order to overcome the paradox presented by Rabin, the following pattern of preferences should be exhibited by the decision maker:

$$(\$110, 0.5; -\$100, 0.5) \prec (\$0, 1), \;\; \forall W < \$300\,000,$$
$$(\$\infty, 0.5; -\$1\,000, 0.5) \succ (\$0, 1), \;\; \forall W > 0.$$

This pattern is exhibited by the decision utility model if the following conditions are satisfied:

$$\begin{aligned} -100 + 210 D^{-1}(0.5) < \ & 0 \\ -1\,000 + \lim_{M\to\infty}(M + 1\,000) D^{-1}(0.5) > \ & 0 \end{aligned} \Rightarrow D\left(\frac{100}{210}\right) > \frac{1}{2}.$$

The resulting condition is satisfied by any decision utility function which is steep enough in the interval $\left[0, \frac{100}{210}\right]$. This is the case for the decision utility function estimated in Appendix 1.

The conclusion of the descriptive part of arguments supporting the decision utility model is that it fits data and predicts some well-known paradoxes using an S-shaped decision utility function instead of probability weighting.

8 Related models

In this section we relate our model with the existing literature. A three criteria decision model of Cohen (1992) is closest to our approach. The model assumes that choices between lotteries depend on the security level and the potential level (i.e. the worst and the best of probable outcomes). For some lotteries choice is represented by the mere comparison of those levels in both lotteries. If this is not sufficient, the choice is completed by the value comparison of an affine function of the expected utility, the coefficients of which depend on both the security and potential level. Let P be a lottery, and x_l, x_u its worst and best outcome. Then the utility of this lottery in Cohen's model is given by

$$V(P) = a(x_l, x_u)\mathbb{E}u(P) + b(x_l, x_u), \tag{7}$$

where u is the standard vNM utility function and $a(x_l, x_u) > 0$, $b(x_l, x_u)$ are coefficients, which depend only on x_l, x_u. The model shares some important characteristics of our approach: Expected Utility for lotteries sharing the same range and discontinuity at the legs in the Marschak-Machina triangle. At the same time there are important differences.

Cohen's model is only slightly less general than the range-dependent utility model and therefore allows a comparable number of free elements: separate coefficients $a(x_l, x_u)$, $b(x_l, x_u)$ for every range in addition to a single utility function u. Cohen did not propose any operational model suitable for prediction, such as the decision utility model. Apart from that there is a number of differences between the two approaches: 1) Lotteries are compared via their utility values calculated according to 7 in Cohen's approach. This creates the issue of noncomparable utility values for different ranges and a bunch of technical issues such as the notions of overlapping and connected preferences. These problems are avoided in our approach where lotteries are compared via their CE values. 2) Cohen explicitly assumes the dominance axiom (her axiom 5) which leads to conditions on the coefficients $a(x_l, x_u), b(x_l, x_u)$; these lack an intuitive interpretation. We provide first the representation and then provide the necessary and sufficient conditions to guarantee the monotonicity in the decision utility model. 3) The axioms of independence and continuity hold only for lotteries having the same range in Cohen's approach; they hold for lotteries comparable within the same range in our approach. Moreover, Cohen needs an additional axiom strengthening independence, that we do not require.

The Lottery Dependent Utility model (Becker and Sarin, 1987), although seemingly related, is further apart from our approach. It departs from the classical EU model by allowing the utility of a given lottery prize to depend upon the attributes of the lottery itself; these attributes are captured by a function h. Additionally, the utility function is assumed to be exponential.

A support for an S-shaped utility function was given by several authors. Markowitz (1952) proposed his hypothesis of a double S-shaped utility function: one for gains, and one for losses. This allows predicting the coexistence of risk aversion and risk seeking both in the gain and in the loss domain. Bordley and LiCalzi (2000) show that the set of Savage (1954) axioms implies that one should select an action which maximizes the probability of meeting an uncertain target. As the utility function U defined to be equal to this probability is bounded and increasing, it has all the properties of a cumulative distribution function (CDF) over consequences. The authors observe that a unimodal probability density for the target corresponds to an S-shaped CDF. In a similar vein Abbas and Matheson (2009) analyze the effects of performance targets on decisions and show that many target-based incentives induce S-shaped utility functions. Rayo and Becker (2007) study hedonic utility and derive two types of happiness functions. The first type is a step function that delivers the maximum level of happiness whenever the agent exceeds his performance benchmark. The second type has a smoother "S" shape and arises when the agent has an information advantage over the principal when selecting his actions.

Besides our model, there are other approaches that use elements of Range-Frequency Theory of Parducci (1965). Stewart et al. (2014) manipulate probabilities and outcomes and find that the estimated shapes of the value and probability weighting functions depend on the entire set of lotteries involved in a given experiment. To explain this, they use the frequency principle of Parduccis theory. The frequency principle is not used in the decision utility model as we regard it as the second order effect. This topic is left for a future version of the theory, in which also the distribution of outcomes (more specifically its skewness) in the lottery range impacts risky decisions.

9 Conclusions

In this paper we have introduced a new model of decision-making under risk. Support for this model can be given using various criteria:

Descriptive accuracy: Most of the existing experimental evidence in decision-making under risk involves binary lotteries. In this case range-dependence is equally supported as rank-dependence. Descriptive predictions of the two concepts differ in the case of multi-outcome lotteries, with a lot of evidence supporting range-dependence as discussed n Section 6.2.

Psychological plausibility: The two concepts offer, however, an entirely different psychological explanation of the same empirical evidence. Unlike its mathematical elegance, psychological plausibility of cumulative probability weighting is disputable (Birnbaum, 2004). Range-dependence, on the other hand, seems more natural - examples supporting this view have been given in Section 1.

Normative appeal: It is well accepted that Expected Utility Theory defines the standard of rationality for decision-making under risk. However the standard EU model fails in accommodating experimental evidence in many decision contexts known as EU paradoxes. The model proposed here aims at explaining these paradoxes by departing from the rationality principle as little as possible. Therefore the decision utility model retains linearity in probabilities and converges to the standard EU model as decision makers enrich the support of the lotteries and make the ranges wider.

Predictive power: The decision utility model gives strong testable predictions because it is refutable and parsimonious: its only free element is the decision utility function. In CPT, by comparison, we have the value function, the probability weighting function, and the location of a reference point. The shortcoming of the decision utility model is, however, that loss aversion is not incorporated. We leave it for future work.

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Appendix 1: Estimation details and data

We provide estimation details of the range-dependent and decision utility models (further RngDU and DU) for the Tversky and Kahneman (1992) data which were discussed in Sections 3.1, 3.2, 6.2 and 7. We demonstrate that the decision utility function D can easily be measured in practical decision analysis applications. As shown below the estimation procedure of the DU model is the same as that of the EU and CPT models.

The (CE values, probability) pairs were fitted using a standard nonlinear least squares procedure implemented with the Wolfram Mathematica® NonlinearModelFit function (similar functions are available in many popular statistical packages – R, Matlab, or MS Excel). The following specifications were used:

1. The **RngDU and DU models** of Theorem 1 and 2 applied to binary lotteries (see equation 6) were fitted using the decision utility function of the form of the CDF of the Two-Sided Power Distribution (see Example 2 in Section 5.2) having two parameters x_0, α:

$$\mathrm{CE}_{\mathrm{RngDU/DU}} = x_l + (x_u - x_l) \times \begin{cases} x_0 \left(\frac{p}{x_0}\right)^{1/\alpha}, & \text{for } 0 \le p \le x_0, \\ 1 - (1 - x_0)\left(\frac{1-p}{1-x_0}\right)^{1/\alpha}, & \text{for } x_0 < p \le 1. \end{cases} \tag{8}$$

   Note that: a) the formula (8) uses the inverse of (5) - the definition of the CDF of the TSPD, b) we assume the same formula (8) to derive a single DU model and many RngDU models (each with its own pair of parameters) c) by using CDF we implicitly impose two restrictions on the estimated utility functions: the value of 0 at the lower end and of 1 at the upper end of the respective domain. This resulted in:

   (a) Seven RngDU models corresponding to seven lottery ranges:

Range	$[0, 50]$	$[0, 100]$	$[0, 200]$	$[0, 400]$	$[50, 100]$	$[50, 150]$	$[100, 200]$
$\hat{x_0}$	.33 (.00)	.25 (.02)	.15 (.08)	0.16 (.00)	0.32 (.06)	0.23 (.02)	0.37 (.04)
$\hat{\alpha}$	2.00 (.01)	2.34 (.10)	2.05 (.23)	1.65 (.00)	2.58 (.39)	2.29 (.12)	2.29 (.23)

   (b) Single DU model in two versions: i) DU: the form specified in (8), which minimizes nominal error – this form was reported in Table 1; or ii) DU': the form, where CE values are normalized (subtract x_l from both sides of (8) and divide them by $x_u - x_l$), which minimizes relative error – this form was used to derive Figure 2. The estimated parameters in these two models were:

Model	DU	DU'
$\hat{x_0}$	.20 (.03)	.27 (.02)
$\hat{\alpha}$	1.94 (.08)	2.18 (.10)

2. **CPT model**: The form and parameter values $\hat{\beta}, \hat{\gamma}$ were taken from Tversky and Kahneman (1992):

$$\text{CE}_{\text{CPT}} = \left[x_l^\beta + \left(x_u^\beta - x_l^\beta \right) \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \right]^{1/\beta} \quad \hat{\beta} = 0.88, \;\; \hat{\gamma} = 0.61.$$

3. **EU model** (of income): The utility function is assumed to be a power function with parameter β:

$$\text{CE}_{\text{EU}} = \left[x_l^\beta + \left(x_u^\beta - x_l^\beta \right) p \right]^{1/\beta} \quad \hat{\beta} = 0.60 \; (.08).$$

The predicted CE values and Sum of Squared Errors of the models are presented in Table 1.

Appendix 2: Proofs

Proof of theorem 1

Step 1. *Necessity: The representation given by (1) satisfies axioms A1–A4.*

It is obvious for Axioms 1 and 4 (Weak Order and Montonicity). Consider Within-Range Continuity. Take any $[x_l, x_u] \subset X$, $x_l < x_u$ and $P, Q, R \in L^c_{[x_l,x_u]}$. By expression (1), the following is true: $P \succ Q \succ R \iff \text{CE}(P) > \text{CE}(Q) > \text{CE}(R)$. Since $u_{[x_l,x_u]}$ is a strictly increasing and surjective function, it follows that $u_{[x_l,x_u]}(\text{CE}(P)) > u_{[x_l,x_u]}(\text{CE}(Q)) > u_{[x_l,x_u]}(\text{CE}(R))$ and: $\lim_{\alpha \to 1} u^{-1}_{[x_l,x_u]}(\alpha u_{[x_l,x_u]}(\text{CE}(P)) + (1-\alpha) u_{[x_l,x_u]}(\text{CE}(R))) = \text{CE}(P)$, $\lim_{\alpha \to 0} u^{-1}_{[x_l,x_u]}(\alpha u_{[x_l,x_u]}(\text{CE}(P)) + (1-\alpha) u_{[x_l,x_u]}(\text{CE}(R))) = \text{CE}(R)$. Therefore there exist $\alpha, \beta \in (0,1)$, such that $u^{-1}_{[x_l,x_u]}(\alpha \text{CE}(P) + (1-\alpha)\text{CE}(R)) > \text{CE}(Q) > u^{-1}_{[x_l,x_u]}(\beta \text{CE}(P) + (1-\beta)\text{CE}(R))$. This in turn is equivalent to $\alpha P + (1-\alpha) R \succ Q \succ \beta P + (1-\beta) R$, and hence Within-Range Continuity is satisfied.

Consider Within-Range Independence. Take any $[x_l, x_u] \subset X$, $x_l < x_u$, any $P, Q \in L^c_{[x_l,x_u]}$ and any $R \in L : \alpha P + (1-\alpha) R, \alpha Q + (1-\alpha) R \in L^c_{[x_l,x_u]}$, for some $\alpha \in (0,1)$. Using (1) and the fact that $u_{[x_l,x_u]}$ is strictly increasing and surjective, we deduce that the axiom must hold: $P \succsim Q \iff \text{CE}(P) \geq \text{CE}(Q) \iff \sum_{x \in \text{supp}(P)} P(x) u_{[x_l,x_u]}(x) \geq \sum_{x \in \text{supp}(Q)} Q(x) u_{[x_l,x_u]}(x) \iff \alpha \sum_{x \in \text{supp}(P)} P(x) u_{[x_l,x_u]}(x) + (1-\alpha) \sum_{x \in \text{supp}(R)} R(x) u_{[x_l,x_u]}(x) \geq \alpha \sum_{x \in \text{supp}(Q)} Q(x) u_{[x_l,x_u]}(x)$ $+ (1-\alpha) \sum_{x \in \text{supp}(R)} R(x) u_{[x_l,x_u]}(x) \iff \sum_{x \in \text{supp}(\alpha P + (1-\alpha) R}(\alpha P + (1-\alpha) R)(x) u_{[x_l,x_u]}(x) \geq \sum_{x \in \text{supp}(\alpha Q + (1-\alpha) R)}(\alpha Q + (1-\alpha) R)(x) u_{[x_l,x_u]}(x) \iff \text{CE}(\alpha P + (1-\alpha) R) \geq \text{CE}(\alpha Q + (1-\alpha) R) \iff \alpha P + (1-\alpha) R \succsim \alpha Q + (1-\alpha) R$.

Step 2. *If a preference relation $\succsim \subset L \times L$ satisfies axioms A1–A4, then for any $\alpha, \beta \in (0,1)$ and $x_l, x_u \in X : x_l < x_u$, the following holds:*

$$\alpha > \beta \iff \alpha P^{x_u} + (1-\alpha) P^{x_l} \succ \beta P^{x_u} + (1-\beta) P^{x_l}. \tag{9}$$

No	x_l	x_u	p	CE	EV	EU	DU	CPT	RngDU
1	0	50	0.10	9.0	5.0	1.1	7.0	7.4	9.0
2	0	50	0.50	21.0	25.0	15.7	18.6	18.7	21.0
3	0	50	0.90	37.0	45.0	41.9	36.3	34.0	37.0
4	0	100	0.05	14.0	5.0	0.7	9.9	10.0	12.5
5	0	100	0.25	25.0	25.0	9.9	22.7	24.6	24.9
6	0	100	0.50	36.0	50.0	31.5	37.3	37.4	36.8
7	0	100	0.75	52.0	75.0	61.9	56.1	52.6	53.0
8	0	100	0.95	78.0	95.0	91.8	80.8	76.9	76.4
9	0	200	0.01	10.0	2.0	0.1	8.6	7.4	8.0
10	0	200	0.10	20.0	20.0	4.3	28.2	29.6	24.6
11	0	200	0.50	76.0	100.0	63.0	74.6	74.8	68.7
12	0	200	0.90	131.0	180.0	167.8	145.2	135.9	140.1
13	0	200	0.99	188.0	198.0	196.7	183.3	180.0	180.5
14	0	400	0.01	12.0	4.0	0.2	17.2	14.9	12.0
15	0	400	0.99	377.0	396.0	393.4	366.5	360.1	377.0
16	50	100	0.10	59.0	55.0	54.4	57.0	59.0	60.1
17	50	100	0.50	71.0	75.0	73.3	68.6	70.5	69.8
18	50	100	0.90	83.0	95.0	94.4	86.3	85.2	83.8
19	50	150	0.05	64.0	55.0	53.9	59.9	62.4	61.7
20	50	150	0.25	72.5	75.0	70.9	72.7	77.7	73.6
21	50	150	0.50	86.0	100.0	94.7	87.3	90.5	86.0
22	50	150	0.75	102.0	125.0	121.1	106.1	105.3	102.8
23	50	150	0.95	128.0	145.0	144.0	130.8	128.4	126.6
24	100	200	0.05	118.0	105.0	104.3	109.9	112.7	115.6
25	100	200	0.25	130.0	125.0	122.4	122.7	128.2	131.4
26	100	200	0.50	141.0	150.0	146.6	137.3	141.1	143.3
27	100	200	0.75	162.0	175.0	172.5	156.1	155.8	158.1
28	100	200	0.95	178.0	195.0	194.4	180.8	178.7	179.2
				SSE	6767.3	4785.5	731.1	656.6	266.2

Table 1: Median (for 25 subjects) Certainty Equivalents for 28 prospects involving gains, taken from Tversky and Kahneman (1992). Each prospect offers x_u with probability p and x_l with the remaining probability. For example median Certainty Equivalent for lottery ($\$\mathbf{0}, \mathbf{0.9}, \$\mathbf{50}, \mathbf{0.1}$) is equal to $\$\mathbf{9}$. Additionally, predicted theoretical values for CE using different estimated models are presented: EV – Expected Value, EU – Expected Utility: one universal fitted utility function in the whole range $[0, 400]$, DU – the decision utility model, CPT – the Cumulative Prospect Theory model, RngDU – the range-dependent utility models. The Sum of Squared Errors (SSE) for each of the model is reported in the last raw.

Proof. First the $\Longrightarrow$ direction is proved. Within-Range Independence implies that the same axiom holds with the weak preference relation $\succsim$ replaced by the strict preference relation $\succ$. Using Restricted Monotonicity and the strict version of Within-Range Independence twice, one gets: $x_u > x_l \iff P^{x_u} \succ P^{x_l} \iff P^{x_u} \succ \beta P^{x_u} + (1-\beta)P^{x_l} \iff \alpha P^{x_u} + (1-\alpha)P^{x_l} = \gamma P^{x_u} + (1-\gamma)(\beta P^{x_u} + (1-\beta)P^{x_l}) \succ \beta P^{x_u} + (1-\beta)P^{x_l}$, where $\gamma = \frac{\alpha-\beta}{1-\beta}$ and the last step follows from $1 > \alpha > \beta \Longrightarrow \gamma \in (0,1)$. Hence the RHS of the equivalence in (9) holds.

Next the $\Longleftarrow$ direction is proved. It is proved by contradiction. Suppose $\alpha \leq \beta$. If $\alpha = \beta$, then the RHS of the equivalence in (9) does not hold. If $\alpha < \beta$, then the same reasoning as above with the roles of α and β reversed implies that $\beta P^{x_u} + (1-\beta)P^{x_l} \succ \alpha P^{x_u} + (1-\alpha)P^{x_l}$, so the RHS of the equivalence in (9) does not hold either. A contradiction. □

Step 3. *A construction of a strictly increasing and surjective mapping* $u_{[x_l,x_u]} : [x_l, x_u] \to [0, 1]$.

For $x \in (x_l, x_u) \subset X$, $x_l < x_u$, consider two sets: $\{\alpha \in (0,1) : \alpha P^{x_u} + (1-\alpha)P^{x_l} \succ P^x\}$, $\{\alpha \in (0,1) : \alpha P^{x_u} + (1-\alpha)P^{x_l} \prec P^x\}$. These sets are nonempty by Restricted Monotonicity and Within-Range Continuity: $x_u > x > x_l \iff P^{x_u} \succ P^x \succ P^{x_l} \implies \exists \alpha, \beta \in (0,1) : \alpha P^{x_u} + (1-\alpha)P^{x_l} \succ P^x \succ \beta P^{x_u} + (1-\beta)P^{x_l}$. They are open by Within-Range Continuity and disjoint by Weak Order. Hence these sets cannot cover $(0,1)$. There must exist $\alpha_x \in (0,1)$ for which $\alpha_x P^{x_u} + (1-\alpha_x)P^{x_l} \sim P^x$. Step 2 implies that $\alpha_x \in (0,1)$ is unique. Suppose it is not. W.l.o.g. let $\alpha_1 > \alpha_2$ such that: $P^x \sim \alpha_1 P^{x_u} + (1-\alpha_1)P^{x_l} \sim \alpha_2 P^{x_u} + (1-\alpha_2)P^{x_l}$. By step 2, it holds that: $\alpha_1 P^{x_u} + (1-\alpha_1)P^{x_l} \succ \alpha_2 P^{x_u} + (1-\alpha_2)P^{x_l}$, which violates transitivity and hence Weak Order. Additionally, $\alpha_{x_l} = 0$ and $\alpha_{x_u} = 1$ since $P^{x_l} \sim \alpha_{x_l} P^{x_u} + (1-\alpha_{x_l})P^{x_l}$ only if $\alpha_{x_l} = 0$ and $P^{x_u} \sim \alpha_{x_u} P^{x_u} + (1-\alpha_{x_u})P^{x_u}$ only if $\alpha_{x_u} = 1$ by step 2.

Define the following mapping $u_{[x_l,x_u]} : [x_l, x_u] \to [0,1]$, such that $u_{[x_l,x_u]}(x) = \alpha_x$, where $P^x \sim \alpha_x P^{x_l} + (1-\alpha_x)P^{x_l}$. By Restricted Monotonicity and Within-Range Continuity (and the reasoning above) this mapping is strictly increasing and surjective. Hence it is also continuous. It follows that there exists an inverse mapping $u^{-1}_{[x_l,x_u]} : [0,1] \to [x_l, x_u]$, which is also strictly increasing and surjective.

Step 4. *The CE representation of a nondegenerate lottery.*

Any nondegenerate lottery $P \in L$ can be written in the following way: $P = p_1 P^{x_1} + p_2 P^{x_2} + ... + p_n P^{x_n}$, where $\{x_1, x_2, ..., x_n\} = \text{supp}(P)$, $x_i < x_j$, for $i < j$, $i, j \in \{1, 2, ..., n\}, i \neq j$, $p_i > 0$, $i \in \{1, 2, ..., n\}$, $\sum_{i=1}^n p_i = 1$ and $n \geq 2$ is a finite natural number. The range of this lottery is equal to $[x_1, x_n]$. By Step 3 for each element of the support $x_i \in \text{supp}(P)$, $i \in \{1, ..., n\}$, there exists a unique number α_i, such that $P^{x_i} \sim \alpha_i P^{x_n} + (1-\alpha_i)\,P^{x_1}$. Note that $\alpha_1 = 0$ and $\alpha_n = 1$. Define a sequence of lotteries $\{P_i\}_{i\in\{0,1,...,n-2\}}$, such that: $P_0 = P$, $P_i = \left(p_1 + \sum_{j=2}^{i+1} p_j(1-\alpha_j)\right) P^{x_1} + \sum_{j=i+2}^{n-1} P^{x_j} + \left(p_n + \sum_{j=2}^{i+1} p_j\alpha_j\right) P^{x_n}$, for $i = 1, 2, ..., n-2$. Observe that $P_{n-2} = \sum_{j=1}^n p_j(1-\alpha_j)P^{x_1} + \sum_{j=1}^n p_j\alpha_j P^{x_n} = \left(1 - \sum_{j=1}^n p_j\alpha_j\right) P^{x_1} + \sum_{j=1}^n p_j\alpha_j P^{x_n}$. Define another sequence of lotteries $\{Q_i\}_{i=0,1,...,n-3}$, such that: $Q_i(x) = 0$, if $x = x_{i+2}$, and $Q_i(x) = \frac{P_i(x)}{1-p_{i+2}}$, if $x \neq x_{i+2}$. The lottery P_i can thus be written as $P_i = p_{i+2}P^{x_{i+2}} + (1-p_{i+2})Q_i$, for $i = 0, 1, ..., n-3$.

Within-Range Independence applied twice (second time with the roles of P, Q switched) implies that the same axiom holds with the weak preference relation $\succsim$ replaced by the indifference relation $\sim$, i.e. for any interval $[x_l, x_u] \subset X$, $x_l < x_u$, for every $R, S, T \in L$, such that $\alpha R + (1-\alpha)T, \alpha S + (1-\alpha)T \in L^c_{[x_l,x_u]}$, for all $\alpha \in (0,1]$ the following holds: $R \sim S \iff \alpha R + (1-\alpha)T \sim \alpha S + (1-\alpha)T$, $\forall \alpha \in [0,1]$. For $i = 0, 1, ..., n-3$, apply this version of the axiom with $\alpha = p_{i+2}$, $R = P^{x_{i+2}}$, $S = (1-\alpha_{i+2})P^{x_1} + \alpha_{i+2}P^{x_n}$, $T = Q_i$, to get $P_i \sim P_{i+1}$. Note that R, S, T satisfy conditions of the axiom, i.e. they are either nondegenerate lotteries with the same range $[x_1, x_n]$ or degenerate lotteries with support in the interval $[x_1, x_n]$. Axiom 1 (transitivity of $\succsim$) applied twice implies that $\sim$ is transitive as well. Applying transitivity of $\sim$ finitely many times we get: $P \equiv P_0 \sim P_1 \sim ... \sim P_{n-2} = (1 - \sum_{i=1}^n p_i\alpha_i)\,P^{x_1} + \sum_{i=1}^n p_i\alpha_i P^{x_n}$. All the lotteries in this sequence have the same range equal to $[x_1, x_n]$. Define a mapping $u_{[x_1,x_n]} : [x_l, x_n] \to [0,1]$, such that: $u_{[x_1,x_n]}(x_i) = \alpha_i$, for $i = 1, 2, ..., n$. Step 3 implies that there exists an inverse mapping $u^{-1}_{[x_1,x_n]}$ and after defining $\text{CE}(P) = u^{-1}_{[x_l,x_u]}\left(\sum_{i=1}^n p_i u_{[x_l,x_u]}(x_i)\right)$ it follows that $P \sim P^{\text{CE}(P)}$.

Step 5. *Two nondegenerate lotteries' comparison via the certainty equivalents.*

By Weak Order and Restricted Monotonicity any two nondegenerate lotteries P_1, P_2 can be compared using CE defined in a previous step: $P_1 \succsim P_2 \iff \text{CE}(P_1) \geq \text{CE}(P_2)$.

Step 6. *The comparison involving degenerate lotteries.*

It remains to be showed, that in case one or two of the lotteries being compared are degenerate, one can replace the corresponding side of the inequality in (1) by its one-element support. If both lotteries are degenerate, i.e. $P = P^x, Q = P^y$, Restricted Monotonicity states that they are represented by their one-element support $P \succ Q \iff x > y$. If only one is degenerate $P = P^x$ and the other nondegenerate $Q \in L \setminus L^d$ with range $[x_l, x_u] \subset X$, $x_l < x_u$, then there are three cases: a) if $x \geq x_u$, then $P^x \succsim P^{x_u} \succ Q$, by Restricted Monotonicity and Independence, and hence $P^x \succ Q$, by Weak Order; b) similarly if $x \leq x_l$, then $P^x \precsim P^{x_l} \prec Q$, and hence $P^x \prec Q$; c) if $x \in (x_l, x_u)$, then $Q \sim P^{\text{CE}(Q)}$, where $\text{CE}(Q)$ is defined as in the RHS of (1), and by Restricted Monotonicity and Weak Order $P^x \succsim Q \iff x \geq \text{CE}(Q)$.

Step 7. *Uniqueness of* $u_{[x_l,x_u]}$.

The argument is the same as in demonstrating cardinal uniqueness of a von Neumann Morgenstern utility function in the Expected Utility Theorem. But since we require that the convex hull of the image of $u_{[x_l,x_u]}$ is equal to $[0, 1]$, then we have uniqueness instead of cardinal uniqueness. This finishes the proof.

Proof of Theorem 2

Step 1. *If axioms 1-4 hold, then Axiom 5 is equivalent to the following condition:*

$$\text{CE}(P_{\alpha,\beta}) = \alpha\text{CE}(P) + \beta,\ \forall P \in L,\ \forall \alpha > 0, \beta \in \mathbb{R}:\ P_{\alpha,\beta} \in L. \tag{10}$$

($\Leftarrow$) Suppose that (10) holds. By the definition of a certainty equivalent $P^{\text{CE}(P)} \sim P$, for any $P \in L$. Hence:

$$\begin{aligned}
P \succsim Q &\iff P^{\text{CE}(P)} \succsim P^{\text{CE}(Q)} && \text{by transitivity of } \succsim \\
&\iff \text{CE}(P) \geq \text{CE}(Q) && \text{by A4} \\
&\iff \alpha\text{CE}(P) + \beta \geq \alpha\text{CE}(Q) + \beta,\ \forall \alpha > 0, \beta \in \mathbb{R} && \\
&\iff \text{CE}(P_{\alpha,\beta}) \geq \text{CE}(Q_{\alpha,\beta}) && \text{by (14)} \\
&\iff P^{\text{CE}(P_{\alpha,\beta})} \succsim P^{\text{CE}(Q_{\alpha,\beta})} && \text{by monotonicity} \\
&\iff P_{\alpha,\beta} \succsim Q_{\alpha,\beta} && \text{by transitivity}
\end{aligned}$$

and hence Axiom 5 holds as well.

($\Rightarrow$): By theorem 1, axioms 1–4 imply that for every interval $[x_l, x_u] \subset X,\ x_l < x_u$ there exists a unique strictly increasing and surjective function $u_{[x_l,x_u]} : [x_l, x_u] \to [0, 1]$, such that for any nondegenerate lottery $P \in L$, the following holds: $P \sim \text{CE}(P)$, where $\text{CE}(P) = u^{-1}_{[x_l,x_u]}\left[\sum_{x \in \text{supp}(P)} P(x) u_{[x_l,x_u]}(x)\right]$. Take any two nondegenerate lotteries with the same range, i.e. $P, Q \in L_{[x_l,x_u]}$, for some $[x_l, x_u] \subset X$,

x_l, x. Since they have the same range and $u_{[x_l,x_u]}$ is strictly increasing, then by representation (1) it holds:

$$P \succsim Q \iff \sum_{x\in \mathrm{supp}(P)} P(x)u_{[x_l,x_u]}(x) \geq \sum_{x\in \mathrm{supp}(Q)} Q(x)u_{[x_l,x_u]}(x) \tag{11}$$

Similarly for any $\alpha > 0$ and $\beta \in \mathbb{R}$ it holds:

$$\begin{aligned} P_{\alpha,\beta} \succsim Q_{\alpha,\beta} &\iff \sum_{x\in \mathrm{supp}(P_{\alpha,\beta})} P_{\alpha,\beta}(x)u_{[\alpha x_l+\beta,\alpha x_u+\beta]}(x) \geq \sum_{x\in \mathrm{supp}(Q_{\alpha,\beta})} Q_{\alpha,\beta}(x)u_{[\alpha x_l+\beta,\alpha x_u+\beta]}(x) \\ &\iff \sum_{x\in \mathrm{supp}(P)} P(x)u_{[\alpha x_l+\beta,\alpha x_u+\beta]}(\alpha x+\beta) \geq \sum_{x\in \mathrm{supp}(Q)} Q(x)u_{[\alpha x_l+\beta,\alpha x_u+\beta]}(\alpha x+\beta) \end{aligned} \tag{12}$$

where equivalence is obtained by changing variables and using the definition of $P_{\alpha,\beta}$, $Q_{\alpha,\beta}$. By axiom 5, it is the case that there is infinitely many utility functions that can be used to represent $P \succsim Q$: not only $u_{[x_l,x_u]}$ as in (11) but also $u_{[\alpha x_l+\beta,\alpha x_u+\beta]}$ for any $\alpha > 0$, $\beta \in \mathbb{R}$ as in (12). By theorem 1 the utility function used to represent lotteries in $L_{[x_l,x_u]}$ is unique. So it must be that:

$$u_{[x_l,x_u]}(x) = u_{[\alpha x_l+\beta,\alpha x_u+\beta]}(\alpha x_l+\beta), \text{ for } x \in [x_l, x_u] \tag{13}$$

It follows that the LHS of (11) and the LHS of (12) must be equal. And the same for RHS. Applying the inverse of $u_{[\alpha x_l+\beta,\alpha x_u+\beta]}$ on the LHS of (12) one obtains $\mathrm{CE}(P_{\alpha,\beta})$, whereas applying the inverse of $u_{[x_l,x_u]}$ on the LHS of (11) results in $\mathrm{CE}(P)$. By (13) it must be the case that $\mathrm{CE}(P_{\alpha,\beta}) = \alpha\mathrm{CE}(P)+\beta$.

Step 2. *Necessity of the axioms.*

The representation given in (2) is a special case of the representation given in (1). Hence it satisfies Axioms 1–4. In what follows we show that it also satisfies axiom 5. Representation (2) implies that:

$$\begin{aligned} \mathrm{CE}(P_{\alpha,\beta}) &= \alpha x_l + \beta + \alpha(x_u - x_l)D^{-1}\left(\sum_{x\in \mathrm{supp}(P_{\alpha,\beta})} P_{\alpha,\beta}(x)D\left(\frac{x-\alpha x_l-\beta}{\alpha(x_u-x_l)}\right)\right) \\ &= \alpha\left[x_l + (x_u-x_l)D^{-1}\left(\sum_{x\in \mathrm{supp}(P)} P_{\alpha,\beta}(\alpha x+\beta)D\left(\frac{x-x_l}{x_u-x_l}\right)\right)\right] + \beta \\ &= \alpha\mathrm{CE}(P)+\beta \end{aligned} \tag{14}$$

where the second equality follows from changing variables and the third from the definition of $P_{\alpha,\beta}$. By direction ($\Leftarrow$) of Step 1 it follows that A5 is satisfied.

Step 3. *Sufficiency of the axioms.*

If one of the lotteries being compared is degenerate then refer to Step 6 of the proof of Theorem 1. In what follows we assume a nondegenerate lottery. Consider lottery P with range equal to $[x_l, x_u] \subset X$, $x_l < x_u$. Define $\alpha = \frac{1}{x_u-x_l}$, $\beta = -\frac{x_l}{x_u-x_l}$. Since it is assumed that $[0,1] \subset X$, so $P_{\alpha,\beta} \in L$. Define the following mapping $D : [0,1] \to [0,1]$, such that $u_{[x_l,x_u]}(x) =: D\left(\frac{x-x_l}{x_u-x_l}\right)$ for all $x \in [x_l, x_u]$. By Theorem 1 and the reasoning in direction $\Rightarrow$ of Step 1 D is unique and by Axiom

5 it my be used to represent preferences over lotteries with any range. Hence: $P_{\alpha,\beta} \sim P^{\mathrm{CE}(P_{\alpha,\beta})}$, where $\mathrm{CE}(P_{\alpha,\beta}) = D^{-1}\left[\sum_{x\in \mathrm{supp}(P_{\alpha,\beta})} P_{\alpha,\beta}(x)D\left(x\right)\right]$. Change the variable under the summation from x to $\alpha x+\beta$ and use the fact that $P_{\alpha,\beta}(\alpha x+\beta) = P(x)$, for all $x \in X$, to get $\mathrm{CE}(P_{\alpha,\beta}) = D^{-1}\left[\sum_{x\in \mathrm{supp}(P)} P(x)D\left(\alpha x+\beta\right)\right]$. On the other hand, by Axiom 5 and Step 1 of the proof, the following holds: $\mathrm{CE}(P) = \frac{1}{\alpha}\mathrm{CE}(P_{\alpha,\beta}) - \frac{\beta}{\alpha} = x_l + (x_u - x_l)D^{-1}\left[\sum_{x\in \mathrm{supp}(P)} P(x)D\left(\frac{x-x_l}{x_u-x_l}\right)\right]$ This finishes the proof.

Proof of Lemma 1

Point a) is proved first. The following differential equation needs to be solved: $-\frac{D''(x)x}{D'(x)} = c$, where c is a constant, and D satisfies $D(0) = 0$ and $D(1) = 1$. The only solution to this equation is $D(x) = x^{\alpha}$, where $\alpha > 0$ and $\alpha := 1 - c$. Point b) is proved next. By point a) above function C takes the following form: $C(x) = x^{\beta}$, where $\beta > 0$. By definition of C, D must be of the form: $D(x) = 1 - (1-x)^{\beta}$, where $\beta > 0$ as claimed.

Proof of Theorem 3

Step 1. *The case of lotteries comparable within one range.*

For any interval $[x_l, x_u] \subset X$, $x_l < x_u$ and for all lotteries in the set $L^c_{[x_l,x_u]}$, the decision utility model with the decision utility function D is equivalent to the expected utility model with a strictly increasing utility function u, such that $u(x) = D\left(\frac{x-x_l}{x_u-x_l}\right)$, $\forall x \in [x_l, x_u]$. As is well known, the Certainty Equivalent functional of this model is monotonic wrt FOSD and continuous. Hence to ensure monotonicity and continuity one needs to check lotteries with differing ranges.

Step 2. *Monotonicity for upward and downward range-changing shift of probability mass.*

Any lottery payoff $\mathbf{y}$ which dominates a given lottery payoff $\mathbf{x}$ wrt FOSD can be constructed from $\mathbf{x}$ by a series of shifts of probability mass from $x \in \mathrm{supp}(\mathbf{x})$ to a point $y > x$. Only shifts for which probability mass is shifted to a point $y > \max\mathrm{supp}(\mathbf{x})$ change the lottery range. Similarly any lottery $\mathbf{y}$ which is dominated by a given lottery payoff $\mathbf{x}$ can be constructed from $\mathbf{x}$ by a series of probability mass shifts from $x \in \mathrm{supp}(\mathbf{x})$ to a point $y < x$. Only shifts for which probability mass is shifted to a point $y < \min\mathrm{supp}(\mathbf{x})$ change the lottery range. By step 1 and the above reasoning, it is sufficient to check that monotonicity is satisfied for such a single probability shift: separately for a downward and separately for an upward shift.

If the original lottery payoff $\mathbf{x}$ is degenerate, then shifting some probability mass upwards results in an increase of the CE value and shifting some probability mass downwards results in a decrease of the CE value. This is due to the fact that $\min\mathrm{supp}(\mathbf{x}) < \mathrm{CE}(\mathbf{x}) < \max\mathrm{supp}(\mathbf{x})$.

Assume that $\mathbf{x}$ is nondegenerate. Consider the set of monetary prizes $x_i \in X$, $i = \{0, 1, ..., n+1\}$, where $n \geq 2$, such that $x_i < x_j$, $\forall i < j$, $i, j \in \{0, 1, ..., n+1\}$. Any nondegenerate lottery payoff $\mathbf{x} \in Lp$ with range equal to $[x_1, x_n]$ can be written as: $\mathbf{x} = (x_1, p_1; x_2, p_2; ...; x_n, p_n)$, such that $p_i > 0$ for all $i \in \{1, 2, ..., n\}$ and $\sum_{i\in\{1,2,...,n\}} p_i = 1$. Define two sequences of lottery payoffs $(\mathbf{x}^{\mathrm{up}}_m)_{m\in\mathbb{N}}$,

$(\mathbf{x}_m^{\mathrm{do}})_{m\in\mathbb{N}}$ as in the following table:

	x_0	x_1	x_2	...	x_n	x_{n+1}	range
$\mathbf{x}$	0	p_1	p_2	...	p_n	0	$[x_1, x_n]$
$\mathbf{x}_m^{\mathrm{up}}$	0	p_1	p_2	...	$\frac{m}{m+1}p_n$	$\frac{1}{m+1}p_n$	$[x_1, x_{n+1}]$
$\mathbf{x}_m^{\mathrm{do}}$	$\frac{1}{m+1}p_1$	$\frac{m}{m+1}p_1$	p_2	...	p_n	0	$[x_0, x_n]$

The sequences represent, respectively, the upward and downward range-changing shift in probability mass constructed from $\mathbf{x}$ which were described above.[9]

Note that $\mathrm{Rng}(\mathbf{x}_m^{\mathrm{up}}) = [x_1, x_{n+1}]$ and $\mathrm{Rng}(\mathbf{x}_m^{\mathrm{do}}) = [x_0, x_n]$. Since for $1 < m' < m''$ it holds that $\mathbf{x}_{m'}^{\mathrm{up}} \succ_{\mathrm{FOSD}} \mathbf{x}_{m''}^{\mathrm{up}}$, therefore by step 1 one obtains $\mathrm{CE}(\mathbf{x}_{m'}^{\mathrm{up}}) > \mathrm{CE}(\mathbf{x}_{m''}^{\mathrm{up}})$. Hence it is both necessary and sufficient to check monotonicity conditions for the limiting case when m tends to ∞, i.e. to check whether the following holds: $\mathrm{CE}(\mathbf{x}) \leq \lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m^{\mathrm{up}})$. Similarly, since for $0 < m' < m''$ it holds that $\mathbf{x}_{m''}^{\mathrm{do}} \succ_{\mathrm{FOSD}} \mathbf{x}_{m'}^{\mathrm{do}}$, therefore by step 1 one obtains $\mathrm{CE}(\mathbf{x}_{m''}^{\mathrm{do}}) > \mathrm{CE}(\mathbf{x}_{m'}^{\mathrm{do}})$. Hence it is both necessary and sufficient to check monotonicity conditions for the limiting case when m tends to ∞, i.e. to check whether the following holds: $\lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m^{\mathrm{do}}) \leq \mathrm{CE}(\mathbf{x})$.

Step 3. *For all nondegenerate lottery payoffs* $\mathbf{x}$ *and the lottery payoff sequences* $(\mathbf{x}_m^{\mathrm{up}})_{m\in\mathbb{N}}, (\mathbf{x}_m^{\mathrm{do}})_{m\in\mathbb{N}}$ *defined in Step 2, the following holds:* $\mathrm{CE}(\mathbf{x}) \leq \lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m^{\mathrm{up}}) \iff \mathrm{RRA_D}(x)$ nondecreasing $\forall x \in [0,1]$, *and* $\lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m^{\mathrm{do}}) \leq \mathrm{CE}(\mathbf{x}) \iff \mathrm{RRA_C}(x)$ nondecreasing $\forall x \in [0,1]$.

The first equivalence is proved first. Let D be the decision utility function representing the decision maker's preferences. Assume that a lottery payoff $\mathbf{x}$ and a prize x_{n+1} are defined as in Step 2 but otherwise are arbitrary. Define two functions $D_1, D_2 : [0,1] \to [0,1]$, such that $D_1(y_i) = D(y_i)$, $D_2(y_i) = D(\lambda y_i)$, where $y_i = \frac{x_i - x_1}{x_n - x_1}$, for $i \in \{1, ..., n\}$ and $\lambda = \frac{x_n - x_1}{x_{n+1} - x_1}$. We can treat these two functions as two vNM utility functions restricted to the interval $[0,1]$. Thus we can apply the Pratt (1964) theorem of comparative relative risk aversion. Since $\lambda \in (0,1)$, the following holds:

$$
\begin{aligned}
& \mathrm{RRA_D}(\cdot) \text{ nondecreasing} \\
\iff & \mathrm{RRA_{D_1}}(x) \geq \mathrm{RRA_{D_2}}(x), \quad \forall x \in [0,1] \\
\iff & D_1^{-1}\left[\mathbb{E}D_1\left(\frac{\mathbf{x} - x_1}{x_n - x_1}\right)\right] \leq D_2^{-1}\left[\mathbb{E}D_2\left(\frac{\mathbf{x} - x_1}{x_n - x_1}\right)\right] \\
\iff & D^{-1}\left[\mathbb{E}D\left(\frac{\mathbf{x} - x_1}{x_n - x_1}\right)\right] \leq \frac{1}{\lambda} D^{-1}\left[\mathbb{E}D\left(\lambda\frac{\mathbf{x} - x_1}{x_n - x_1}\right)\right] \\
\iff & x_1 + (x_n - x_1)D^{-1}\left[\mathbb{E}D\left(\frac{\mathbf{x} - x_1}{x_n - x_1}\right)\right] \leq x_1 + (x_{n+1} - x_1)D^{-1}\left[\mathbb{E}D\left(\frac{\mathbf{x} - x_1}{x_{n+1} - x_1}\right)\right] \\
\iff & \mathrm{CE}(\mathbf{x}) \leq \lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m^{\mathrm{up}})
\end{aligned}
$$

where the second equivalence is implied by Pratt (1964) theorem of comparative relative risk aversion, the third by the definition of D_1, D_2, and the last one by the definition of $(\mathbf{x}_m^{\mathrm{up}})_{m\in\mathbb{N}}$.

The second equivalence is proved similarly. Assume that a lottery payoff $\mathbf{x}$ and a prize x_0 are defined as in Step 2 but otherwise are arbitrary. Define two functions $C_1, C_2 : [0,1] \to [0,1]$, such that:

[9] For the argument presented here, it does not matter from which element of the support of $\mathbf{x}$ the probability mass is taken away to be shifted upwards or downwards. It is assumed that they are taken away from x_n in the case of an upward shift and from x_1 in the case of a downward shift.

$C_1(y_i) = C(y_i)$, $C_2(y_i) = C(\lambda y_i)$, where $y_i = \frac{x_n - x_i}{x_n - x_1}$, for $i \in \{1, ..., n\}$, and $\lambda = \frac{x_n - x_1}{x_n - x_0}$. We can treat these two functions as two vNM utility functions restricted to the interval $[0, 1]$. Since $\lambda \in (0, 1)$, the following holds:

$$\begin{aligned}
& \mathrm{RRA_C}(\cdot) \text{ nondecreasing} \\
\iff & \mathrm{RRA_{C_1}}(x) \geq \mathrm{RRA_{C_2}}(x), \quad \forall x \in [0,1] \\
\iff & C_1^{-1}\left[\mathbb{E}C_1\left(\frac{x_n - \mathbf{x}}{x_n - x_1}\right)\right] \leq C_2^{-1}\left[\mathbb{E}C_2\left(\frac{x_n - \mathbf{x}}{x_n - x_1}\right)\right] \\
\iff & C^{-1}\left[\mathbb{E}C\left(\frac{x_n - \mathbf{x}}{x_n - x_1}\right)\right] \leq \frac{1}{\lambda}C^{-1}\left[\mathbb{E}C\left(\lambda\frac{x_n - \mathbf{x}}{x_n - x_1}\right)\right] = \frac{x_n - x_0}{x_n - x_1}C^{-1}\left[\mathbb{E}C\left(\frac{x_n - \mathbf{x}}{x_n - x_0}\right)\right] \\
\iff & x_n - (x_n - x_0)C^{-1}\left[\mathbb{E}C\left(\frac{x_n - \mathbf{x}}{x_n - x_0}\right)\right] \leq x_n - (x_n - x_1)C^{-1}\left[\mathbb{E}C\left(\frac{x_n - \mathbf{x}}{x_n - x_1}\right)\right] \\
\iff & x_0 + (x_n - x_0)D^{-1}\left[\mathbb{E}D\left(\frac{\mathbf{x} - x_0}{x_n - x_0}\right)\right] \leq x_1 + (x_n - x_1)D^{-1}\left[\mathbb{E}D\left(\frac{\mathbf{x} - x_1}{x_n - x_1}\right)\right] \\
\iff & \lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m^{\mathrm{do}}) \leq \mathrm{CE}(\mathbf{x})
\end{aligned}$$

where the second equivalence is implied by Pratt (1964) theorem of comparative relative risk aversion, the third by the definition of C_1, C_2, the fifth by condition (3), and the last one by the definition of $(\mathbf{x}_m^{\mathrm{do}})_{m\in\mathbb{N}}$. This finishes the proof of monotonicity.

Step 4. *Continuity for lotteries which are not comparable within the same range.*

Consider any lottery payoff $\mathbf{x} \in Lp$ with the distribution $P \in L$. Define a sequence of lottery payoffs $(\mathbf{x}_m)_{m\in\mathbb{N}}$, where each lottery payoff $\mathbf{x}_m \in Lp$ has distribution $P_m \in L$, $m \in \mathbb{N}$, such that $\mathbf{x}_m \xrightarrow{d} \mathbf{x}$. Define: $\lim_{m\to\infty} \min \mathrm{supp}(\mathbf{x}_m) = x'_l$ and $\lim_{m\to\infty} \max \mathrm{supp}(\mathbf{x}_m) = x'_u$. If lottery payoff $\mathbf{x}$ is degenerate with $P(x^*) = 1$, for some $x^* \in \mathbb{R}$, then $x^* \in [x'_l, x'_u]$ and therefore: $\lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m) = x'_l + (x'_u - x'_l)D^{-1}\left(D\left(\frac{x^* - x'_l}{x'_u - x'_l}\right)\right) = x^* = \mathrm{CE}(\mathbf{x})$, and hence for such lottery payoffs continuity is satisfied.

If lottery payoff $\mathbf{x}$ is nondegenerate, there are four cases to be considered, depending on the change of range of the limit of the sequence of lottery payoffs relative to the range of $\mathbf{x}$:

a) $x'_l = x_l$, $x'_u = x_u$ (the range remains the same): Then by step 1 the CE functional is continuous, i.e. $\lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m) = \mathrm{CE}(\mathbf{x})$.

b) $x'_l = x_l$, $x'_u > x_u$ (upward range change): Then since $\mathbf{x}_m$ converges in distribution to $\mathbf{x}$, the following holds: $\lim_{m\to\infty} P_m(x) = P(x)$, for all $x \in \mathrm{supp}(\mathbf{x})$ and $\lim_{m\to\infty} P_m(x) = 0$, for all $x \notin \mathrm{supp}(\mathbf{x})$. It follows that: $\mathrm{CE}(\mathbf{x}) = x_l + (x_u - x_l)D^{-1}\left(\sum_{x\in\mathrm{supp}(\mathbf{x})} P(x)D\left(\frac{x - x_l}{x_u - x_l}\right)\right)$, and: $\lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m) = x_l + (x'_u - x_l)D^{-1}\left(\sum_{x\in\mathrm{supp}(\mathbf{x})} P(x)D\left(\frac{x - x_l}{x'_u - x_l}\right)\right) = x_l + (x_u - x_l)\frac{1}{\lambda} D^{-1}\left(\sum_{x\in\mathrm{supp}(\mathbf{x})} P(x)D\left(\lambda\frac{x - x_l}{x_u - x_l}\right)\right)$, where $\lambda = \frac{x_u - x_l}{x'_u - x_l} \in (0, 1)$. Define $\mathbf{r} = \frac{\mathbf{x} - x_l}{x_u - x_l}$ and suppose the vNM utility function on the set of such relative lotteries is D. Then $\lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m) = \mathrm{CE}(\mathbf{x})$ if and only if $\mathrm{CE}(\lambda\mathbf{r}) = \lambda\mathrm{CE}(\mathbf{r})$ for $\lambda > 0$. By Pratt (1964) theorem of comparative relative risk aversion the latter holds true if and only if the utility function D exhibits CRRA. By Lemma 1 this is the case iff $D(x) = x^\alpha$, where $\alpha > 0$ for $x \in [0, 1]$.

c) $x'_l < x_l$, $x'_u = x_u$ (downward range change): This time it is more convenient to use the CE representation via function C, which is given by (3). Then by a similar argument as in b) the following holds: $\lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m) = x_u - (x_u - x_l')C^{-1}\left(\sum_{x\in\mathrm{supp}(\mathbf{x})} P(x)C\left(\frac{x_u - x}{x_u - x_l'}\right)\right) = x_u - (x_u - x_l)\frac{1}{\gamma}C^{-1}\left(\sum_{x\in\mathrm{supp}(\mathbf{x})} P(x)C\left(\gamma\frac{x_u - x}{x_u - x_l}\right)\right)$, where $\gamma = \frac{x_u - x_l}{x_u - x_l'} \in (0, 1)$. Similarly as in step b), define $\mathbf{r} = \frac{\mathbf{x} - x_l}{x_u - x_l}$ and suppose the vNM utility function on the set of such relative lotteries is C. Then $\lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m) = \mathrm{CE}(\mathbf{x})$ if and only if $\mathrm{CE}(\gamma\mathbf{r}) = \gamma\mathrm{CE}(\mathbf{r})$ for $\gamma > 0$. By Pratt (1964) theorem of comparative relative risk aversion the latter holds true if and only if the utility function C exhibits CRRA. By Lemma 1 this is the case iff $C(x) = x^\alpha$, where $\alpha > 0$ for $x \in [0, 1]$. This is, in turn, equivalent to: $D(x) = 1 - (1 - x)^\alpha$, $\alpha > 0$ for $x \in [0, 1]$.

d) $x_l' < x_l$, $x_u' > x_u$ (upward and downward range change): Then by a similar argument as in b), the following holds: $\lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m) = x_l' + (x_u' - x_l')D^{-1}\left(\sum_{x\in\mathrm{supp}(\mathbf{x})} P(x)D\left(\frac{x - x_l'}{x_u' - x_l'}\right)\right) = x_l + (x_u - x_l)\left[\frac{1}{\alpha}D^{-1}\left(\sum_{x\in\mathrm{supp}(\mathbf{x})} P(x)D\left(\alpha\frac{x - x_l}{x_u - x_l} + \beta\right)\right) - \frac{\beta}{\alpha}\right]$, where $\alpha = \frac{x_u - x_l}{x_u' - x_l'} \in (0, 1)$ and $\beta = \frac{x_l - x_l'}{x_u' - x_l'} > 0$. Similarly as in step b), define $\mathbf{r} = \frac{\mathbf{x} - x_l}{x_u - x_l}$ and suppose the vNM utility function on the set of such relative lotteries is D. Then $\lim_{m\to\infty} \mathrm{CE}(\mathbf{x}_m) = \mathrm{CE}(\mathbf{x})$ if and only if $\mathrm{CE}(\alpha\mathbf{r} + \beta) = \alpha\mathrm{CE}(\mathbf{r}) + \beta$ for $\alpha > 0,\ \beta \in \mathbb{R}$. By Pratt (1964) theorem of comparative relative risk aversion the latter holds true if and only if the utility function D exhibits CARA and CRRA at the same time. This is, in turn, equivalent to D being linear, i.e. $D(x) = x$, for $x \in [0, 1]$.

This finishes the proof of continuity.