Carpooling - motivating examples

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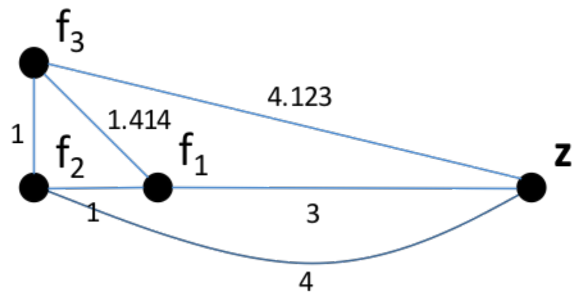
1 Example

Consider the problem described by Kalczynski and Miklas-Kalczynska (2015). Let $F = f_1, f_2, f_3$ be the set of three families and S the algebra of all subsets of F. Let Z be a school.



Define a set-valued characteristic function $v : S \to \mathbb{R}$ with the following values (we assume that the goal is to minimize this value for each player and coalition):

Coalition	Value
$\emptyset$	0
$\{1\}$	6
$\{2\}$	8
$\{3\}$	8.25
$\{1, 2\}$	8
$\{1, 3\}$	8.54
$\{2, 3\}$	9.12
$\{1, 2, 3\}$	15.12

For the carpooling problem this function is sub-additive because for any two subsets of the three families $A, B \in S$, such that $A \cap B = \emptyset$ the following holds

$v(A \cup B) \leq v(A) + v(B)$. A coalition A is beneficial for its members iff for all families $f \in A$ the following holds $v(A) \leq \sum_{f \in A} v(f)$. We may expect that only those coalitions may form that are strictly beneficial to all its members (the condition is satisfied with strict inequality) The value $v(\{1,2,3\})$ comes from the centralized solution in which carpool $\{2,3\}$ is formed and family 1 is left alone. The fact that $v(\{2,3\}) + v(\{1\}) = v(\{1,2,3\})$ reflects the fact that it is not profitable to form three families carpools. This may be due to whatever reason, e.g. it might be impossible to form 3-family carpools. We assume transferable utility, which means that a coalition value may be transferred between its members.

1.1 Dividing coalition value between its members

Suppose that families 2 and 3 form a carpool and family 1 is left alone. We will assume that the value of 9.12 may be split between its members in an arbitrary fashion: e.g. y goes to family 2 and the remaining part $9.12 - y$ goes to family 3. How to implement it in practice? If planning horizon is long enough, it is very easy. Let $n, m \in \mathbb{N}$ be the number of times each family in the carpool $\{2,3\}$ drives. If we want to implement a given division of value $(y, 9.12 - y)$ between the carpool members it is sufficient to set $y = \frac{n}{n+m} \times 9.12$. In this case the average number of miles in this carpool is $\frac{n}{n+m} \times 9.12$ for family 2 and $\frac{m}{n+m} \times 9.12$ for family 3. Any partition of the three families will be called a solution. For example $(\{1\},\{2,3\})$ is a solution in which carpool $\{2,3\}$ is formed and family 1 is left alone. The division of miles for each family in a given solution will be denoted as a triple $(x_1, x_2, x_3) \in \mathbb{R}^3$. Let us first consider the case in which only up to 2 family carpools may be formed.

Solution	Division of miles
$(\{1,2\},\{3\})$	$(x, 8-x, 8.25)$
$(\{1\},\{2,3\})$	$(6, y, 9.12-y)$
$(\{1,3\},\{2\})$	$(8.54-z, 8, z)$

where $(x, y, z) \in \mathbb{R}^3$.

The division of miles between members of a carpool has to be individually rational for members of a carpool, i.e. participating in a carpool has to be profitable for its members. This leads to the following conditions:

$$\begin{aligned} x &\leq 6 \\ 8 - x &\leq 8 \\ y &\leq 8 \\ 9.12 - y &\leq 8.25 \\ z &\leq 8.25 \\ 8.54 - z &\leq 6 \end{aligned}$$

Or after rearranging: $x \in [0, 6]$, $y \in [0.87, 8]$, $z \in [2.54, 8.25]$. (For example $x \leq 6$ means that miles driven by family 1 in the carpool $\{1, 2\}$ cannot exceed miles driven by this family if it stays out of the carpool) We will refer to the above conditions as to individual rationality conditions.

1.2 Standard stability notion

There is no stable solution to this problem no matter which carpool will be formed and no matter how the coalition value will be split among its members. Consider for example the following solution $(\{1\}, \{2, 3\})$ with the division of miles $(6, y, 9.12 - y)$, where $y \in [0.87, 8]$, respecting individual rationality. Suppose that family 1 proposes to form a carpool with family 2. This will be profitable for both families if the resulting division of miles $(x, 8 - x, 8.25)$ satisfies $x \leq 6$ and $8 - x \leq y$. It is possible to find an x satisfying both of these conditions (i.e. $8 - y \leq x \leq 6$) iff $y \in [2, 8]$. Family 1 may alternatively propose to family 3 to form a carpool and leave family 2 alone. It will be profitable for both families if the resulting division of miles $(8.54 - z, 8, z)$ satisfies $8.54 - z \leq 6$ and $z \leq 9.12 - y$. It is possible to find a z satisfying both of these conditions (i.e. $2.54 \leq z \leq 9.12 - y$) iff $y \in [0.87, 6.58]$. The solution $(\{1, 2\}, \{3\})$ blocks the solution $(\{1\}, \{2, 3\})$ iff $y \in [2, 8]$ and the solution $(\{1, 3\}, \{2\})$ blocks the solution $(\{1\}, \{2, 3\})$ iff $y \in [0.87, 6.58]$. We conclude that for $y \in [0.87, 2)$: the latter blocks but not the former, for $y \in [2, 6.58)$: both block and for $y \in [6.58, 8]$: the former blocks but not the latter. In all three cases there exist at least one solution which blocks $(\{1\}, \{2, 3\})$. We conclude that the solution $(\{1\}, \{2, 3\})$ is not stable. One can easily verify that it is also the case for other possible solutions. There does not exists a stable solution in this game according to the standard notion of stability.

1.3 Generalized stability

We propose a generalized notion of stability and show that there exists a stable solution in the game according to the new notion. Continuing the above example, consider the three possible solutions: $(\{1,2\},\{3\})$, $(\{1\},\{2,3\})$, $(\{1,3\},\{2\})$ with the following divisions of miles: $(x, 8-x, 8.25)$, $(6, y, 9.12-y)$, $(8.54-z, 8, z)$, respectively. We require that they are strictly profitable for each carpool member, i.e. individually rational ($x \in [0,6]$, $y \in [0.87, 8]$, $z \in [2.54, 8.25]$).

Suppose now that no matter in which carpool I participate in I get the same average payoff. When I get the same payoff I shall be indifferent between participating in any of them. We want to find $(x,y,z) \in \mathbb{R}^3$ such that the following holds:

$$x = 8.54 - z$$
$$y = 8 - x$$
$$z = 9.12 - y$$

The solution to this problem is $x = 3.71$, $y = 4.29$, $z = 4.83$ and satisfies individual rationality conditions. The following is the division of miles between families provided they are in a carpool:

Solution	Division of miles
$(\{1,2\},\{3\})$	$(3.71, 4.29, 8.25)$
$(\{1\},\{2,3\})$	$(6.00, 4.29, 4.83)$
$(\{1,3\},\{2\})$	$(3.71, 8.00, 4.83)$

We claim that each of the above solutions should be played one third of the time (it is a mixed strategy) to ensure stability which I describe below: Suppose that the solution $(\{1,2\},\{3\})$ is realized. Family 3 may either block it by forming a carpool with family 1 or 2. It has to propose at most 3.71 miles to family 1 or at most 4.29 miles to family 2. In the limit (if they get to drive exactly 3.71 or 4.29 miles) family 3 will drive the same amount of miles in both cases: $8.54 - 3.71 = 4.83$ miles if it forms a carpool with family 1 and $9.12 - 4.29 = 4.83$ miles if it forms a carpool with family 2. Hence family 3 will be indifferent between blocking the solution $(\{1,2\},\{3\})$ with either $(\{1,3\},\{2\})$ or $(\{1\},\{2,3\})$. The same reasoning applies to any other solution. So each of these solutions should be played with probability $\frac{1}{3}$. The mixed strategy solution we propose is written as:

$$\frac{1}{3} \times (\{1,2\},\{3\}) + \frac{1}{3} \times (\{1,3\},\{2\}) + \frac{1}{3} \times (\{1\},\{2,3\})$$

And the amount driven by each family at this solution is equal to:

$$\frac{1}{3}\times(3.71,4.29,8.25)+\frac{1}{3}\times(3.71,8,4.83)+\frac{1}{3}\times(6,4.29,4.83)=(4.47,5.53,5.97)$$

How to interpret this solution? On the first day family 1 goes alone and families 2 and 3 together. On the second day family 2 goes alone and families 1 and 3 together. On the third day family 3 goes alone and families 1 and 2 together. After that we repeat the process. Out of all these days in which families 2 and 3 are in a carpool family 2 drives $\frac{4.29}{9.12}\approx 47.0\%$ of the time and family 3 drives the remaining $\frac{4.83}{9.12}\approx 53.0\%$ of the time. Out of all these days in which families 1 and 3 are in a carpool family 1 drives $\frac{3.71}{8.54}\approx 43,4\%$ of the time and family 3 drives the remaining $\frac{4.83}{8.54}\approx 56,6\%$ of the time. Out of all these days in which families 1 and 2 are in a carpool family 1 drives $\frac{3.71}{8.00}\approx 46.4\%$ of the time and family 2 drives the remaining $\frac{4.29}{8.00}\approx 53.6\%$ of the time. For example if there are 3000 days:

Family	$(\{1,2\},\{3\})$	$(\{1,3\},\{2\})$	$(\{1\},\{2,3\})$	Altogether
1	464	434	1000	1898
2	536	1000	470	2006
3	1000	566	530	2096

The solution presented above is stable also in the following sense. I could write a computer program in which starting with any solution and any division of value for this solution and let families successively block the current solution with their proposals satisfying individual rationality (they want to participate in a carpool) and incentive compatibility (a proposed carpool and the division of miles has to be better for the families in this carpool than the payoff these families received in the previous solution and division of miles). Such a process necessarily leads to one of the three solutions and division of miles specified in the table above. Once reached, the procedure will circle around those three solutions and those division of miles. That means that it is a stable mixed strategy solution. This approach may be generalized.

2 How to model it in general?

Crucial setup questions: Which carpools are feasible? Which carpools are ex ante profitable for their members?

Crucial core questions: Given a structure determined above, what is the appropriate notion of carpool stability? Which carpools (and division of miles inside it) is stable according to this notion?

I think the idea of Aumann Maschler stable sets is the best way to model stability of carpools. I introduce it here in the form of a couple of motivating examples.

Consider a game in characteristic function form (N, w), where N is a set of n families/locations (points in Euclidean space). The subsets of N are called carpools. The (set) value function w is defined in the following way: $w(S)$ equals the negative of the one-time distance traveled by the carpool S. The game (N, w) is equivalent to the game (N, v), such that:

$$v(S) = w(S) + \sum_{i \in S}(-w(i)), \quad \forall S \subseteq N$$

The interpretation of the value $v(S)$ is the overall mile savings of carpool S.

For a game (N, v) a carpool structure is a partition $N = S_1 \cup S_2 \cup ... \cup S_k$, where $k \leq n$ and S_j are disjoint carpools. A payoff n-tuple $\mathbf{x} = (x_1, x_2, ..., x_n)$ (the division of mile savings) is rational for this carpool structure if the following conditions hold:

$$x_i \geq v(i), \quad \forall i \in N$$
$$\sum_{i \in S_j} x_i = v(S_j), \quad \forall j$$

Consider 2 families A and B in the same carpool S_j (we don't allow families from different carpools to object against each other). We say that A **has an objection against** B if there is some carpool $S \subseteq N$ and payoff n-tuple $\mathbf{y} = (y_1, y_2, ..., y_n)$, such that:

O1) $A \in S \not\ni B$

O2) $y_k > x_k, \quad \forall k \in S$

O3) $\sum_{k \in S} y_k = v(S)$

We say that B has a valid counter-objection against A if there is some other coalition T and payoff n-tuple $\mathbf{z} = (z_1, z_2, ..., z_n)$, such that:

CO1) $B \in T \not\ni A$

CO2) $z_k \geq x_k, \quad \forall k \in T$

CO3) $z_k \geq y_K, \quad \forall k \in T \cap S$

CO4) $\sum_{k \in T} z_k = v(T)$

We say $\mathbf{x}$ is stable for the carpool structure $S_1, S_2, ..., S_k$ if every objection can be met by a valid counter-objection. The set of all stable rational n-tuples for the carpool structure is the Aumann-Maschler bargaining set for that structure. Aumann-Maschler proved that any carpool structure for a game in characteristic function form must have at least one stable rational payoff n-tuple.

Example 1. Suppose $N = \{A, B, C\}$, $v(A) = v(B) = v(C) = 0$ and $v(AB) = 60$, $v(AC) = 80$, $v(BC) = 100$, $v(ABC) = 105$.

Consider allocation $(0, 50, 50)$. Family C has an objection against B because there exists a carpool AC which contains C but not B and payoff tuple $(25, 0, 55)$. Family B does not have a valid counter-objection against C because if it matched the offer given to A (25), it would be left with 35, which is less than its original share of 50.

Consider allocation $(0, 40, 60)$. Family C has an objection against B because there is carpool AC and payoff tuple $(15, 0, 65)$. Now family B has a valid counter-objection because there is carpool AB and payoff tuple $(15, 45, 0)$, which is better for B than its original share.

Finding the bargaining set for coalition structures is easy. Suppose the payoff tuple is (x, y, z). Consider different carpool structures:

(AB, C): $80 - x = 100 - y$, and $x + y = 60$, solution: $(20, 40, 0)$

(AC, B): $60 - x = 100 - z$, and $x + z = 80$, solution: $(20, 0, 60)$

(BC, A): $60 - y = 80 - z$, and $y + z = 100$, solution: $(0, 40, 60)$

(ABC): $80 - y = 60 - x = 100 - z$, and $x + y + z = 105$, solution: $(15, 35, 55)$.

Let's make it a little more general:

Example 2. Suppose $N = \{A, B, C\}$, $v(A) = v(B) = v(C) = 0$ and $v(AB) = a$, $v(BC) = b$, $v(AC) = c$, where $a, b, c \geq 0$. Assume that the triangle inequality holds, i.e.

$$a \leq b + c$$
$$b \leq a + c$$
$$c \leq a + b$$

Suppose family A gets α, then family B gets $a - \alpha$ and is willing to pay family C at most $b - a + \alpha$. Family C will be willing to pay family A at most $c - (b - a + \alpha)$. So if $\alpha > c - (b - a + \alpha)$, then family A prefers carpool AB. Otherwise, when $\alpha < c - (b - a + \alpha)$, family A prefers carpool AC. Family A is indifferent if

$\alpha = \frac{a+c-b}{2}$. Similarly one can find the shares of family B and C (β and γ, respectively) in the bargaining set carpools. We shall denote by $(x_A, x_B, x_C; \sigma)$ the elements in the bargaining set, where x_i are shares of mile savings for family i, and σ is a partition of N, which denotes the carpool structure for these shares. The bargaining set is thus:

$$\begin{array}{lrrrll}
(& 0, & 0, & 0; & \{A, B, C\} &) \\
(& \frac{a+c-b}{2}, & \frac{a+b-c}{2}, & 0; & \{AB, C\} &) \\
(& \frac{a+c-b}{2}, & 0, & \frac{c+b-a}{2}; & \{AC, B\} &) \\
(& 0, & \frac{a+b-c}{2}, & \frac{c+b-a}{2}; & \{BC, A\} &)
\end{array}$$

Let's add the carpool ABC with value $v(ABC) = d \geq 0$ to the game. Then the bargaining set will contain the same elements as above *plus* the following additional element:

$$\left(\frac{d-c-b+2a}{3}, \quad \frac{d-a-b+2c}{3}, \quad \frac{d-a-c+2b}{3}; \quad \{ABC\} \right)$$

Let's now analyze an example with four families and possible carpools of three families.

Example 3. Suppose $N = \{A, B, C, D\}$, $v(A) = v(B) = v(C) = v(D) = 0$ and $v(ABC) = a$, $v(ABD) = b$, $v(ACD) = c$, $v(BCD) = d$, where $a, b, c, d \geq 0$. Assume that the following inequalities hold, i.e.

$$2a \leq b + c + d$$
$$2b \leq a + c + d$$
$$2c \leq a + b + d$$
$$2d \leq a + b + c$$

These inequalities are the generalized version of the triangle inequality. It follows from them that any three of the numbers a, b, c, d satisfy the triangle inequality. Moreover, an equality $a = b + c$ can occur only when $a = d$.

Suppose that families A and B get α and β respectively. Then family C gets $a - \alpha - \beta$ in ABC. Then family D will get $b - \alpha - \beta$ in the carpool ABD, $c - a + \beta$ in the carpool ACD and $d - a + \alpha$ in the carpool BCD. In order that no carpool is better than others (meaning that it could exert threats on others), it is necessary and sufficient that:

$$b - \alpha - \beta = c - a + \beta = d - a + \alpha$$

Hence $\alpha = \frac{a+b+c-2d}{3}$ and $\beta = \frac{a+b+d-2c}{3}$. The same argument can be applied to any other pair of families. As a result we get the bargaining set:

$$
\begin{array}{llllll}
(& 0, & 0, & 0, & 0; & \{A,B,C,D\} \quad) \\
(& \frac{a+b+c-2d}{3}, & \frac{a+b+d-2c}{3}, & \frac{a+c+d-2b}{3}, & 0; & \{ABC,D\} \quad) \\
(& \frac{a+b+c-2d}{3}, & \frac{a+b+d-2c}{3}, & 0, & \frac{b+c+d-2a}{3}; & \{ABD,C\} \quad) \\
(& \frac{a+b+c-2d}{3}, & 0, & \frac{a+c+d-2b}{3}, & \frac{b+c+d-2a}{3}; & \{ACD,B\} \quad) \\
(& 0, & \frac{a+b+d-2c}{3}, & \frac{a+c+d-2b}{3}, & \frac{b+c+d-2a}{3}; & \{BCD,A\} \quad)
\end{array}
$$

Looking at the result of both examples one can derive the following *ad hoc* rule, which perhaps works more generally: the value of each carpool is equally divided among its members. If a family enters a carpool she gets the sum of "her shares" minus the sum of the "shares", which her partners get from carpools which do not include her. For example, in the last example above, the family A shares are $\frac{a}{3}$, $\frac{b}{3}$, and $\frac{c}{3}$. Her partners from carpool ABC (B and C) have shares equal to $\frac{d}{3}$ each from carpool BCD, which does not include A. Hence family A gets $\frac{a+b+c-2d}{3}$ if she enters carpool ABC.

The next example concerns 4 families and possible carpools of 2 families.

Example 4. Suppose $N = \{A, B, C, D\}$, $v(A) = v(B) = v(C) = v(D) = 0$ and $v(AB) = a$, $v(BC) = b$, $v(CD) = c$, $v(AC) = d$, $v(BD) = e$, $v(AD) = f$ where $a, b, c, d, e, f \geq 0$. Let $a + c \geq d + e$ and $a + c \geq b + f$ meaning that among three relevant carpool structures (AB, CD), (AC, BD) and (AD, BC), the first one has maximal overall miles' savings. Then it can be proved that there always exist an allocation (x_A, x_B, x_C, x_D) for the structure (AB, CD) which belongs to the bargaining set.

Crucial questions:

- When carpools of at most 2 families are allowed: we know how to solve it when there are 3 or 4 families in total (Example 2 and 4). Is it possible to generalize it for the case of more families in total. Is there a good way to divide all the families into groups of 3-4 families, within which we could use the existing solutions? (By good I mean justified with some reasonable stability criteria.) This would decompose a problem into a series of smaller problems and avoid the issue of "combinatorial expansion"?

- When carpools of at most 3 families are allowed: in what way should we change the analysis of Example 3 to get the solution. And then, the same issue as in the bullet above.

References

Kalczynski, P. and M. Miklas-Kalczynska (2015). A decentralized solution to the youth car pooling problem.