1 Sources of uncertainty and the ambiguity triangle

2 Setup

There is an outcome space $X \subseteq \mathbb{R}$ and a set of states S. We assume that S covers all possible sources of uncertainty. An event E is any subset of S. Let $\mathcal{S}$ be a σ-algebra of events. An act is a mapping $f : S \to X$. The set of all acts is denoted by F. There is a (binary) preference relation $\succsim$ over the set of acts.

In this paper we focus on the following simple setup. An urn with white and black balls inside is given. We assume that $\mathbb{N}$ is the set of natural numbers containing zero. Let $n \in \mathbb{N}$ be the number of all balls in the urn. Let $b, w \in \mathbb{N}$, such that $b + w \leq n$ be the number of known black and white balls in the urn, respectively. Let $W_{(b,w)} \in \mathcal{S}$ be the event of drawing a white a ball from an urn with b known black balls and w known white balls. Similarly let $B_{(b,w)} \in \mathcal{S}$ be the event of drawing a white ball from an urn with b known black balls and w known white balls. We assume after Chew, Sagi (2008) that a given source of uncertainty is a collection of events which forms a partition of a state space. From now on we assume that the overall number of balls n in the urn is fixed. Thus $\{W_{(b,w)}, B_{(b,w)}\}$, where $b, w \in \mathbb{N}$, such that $b + w \leq n$ is a source of uncertainty. The set of all sources of uncertainty in our setup will be denoted by $\mathcal{U}$. We thus have $\frac{1}{2}(n+1)^2$ number of possible sources of uncertainty.

Given a source of uncertainty $\{W_{(b,w)}, B_{(b,w)}\} \in \mathcal{U}$ we will denote a typical (ambiguous) prospect as $(x, W_{(b,w)}; y, B_{(b,w)})$, where $x, y \in X$. The interpretation is the following: if I draw a white ball from the urn containing n white or black balls in which I know that b are black and w are white and the colors of the remaining balls are unknown to me then I get x as a prize. If from the same urn I draw a black ball, then I get y as a prize. Each source of uncertainty corresponds to a different ambiguous prospect. A prospect is defined as an urn with n balls overall, in which the number w of white and the number b of black balls are known to a decision maker, while whether the color of each of the $n - b - w$ remaining balls is white or black remains unknown.

For a fixed overall number of balls in the urn n we define an ambiguity triangle as the set of all sources of uncertainty (ambiguous prospects) in the possible (b, w) space, i.e.

$$\Delta_A = \{(b, w) \in \mathbb{N}^2 | b + w \leq n\}$$

We can normalize the ambiguity triangle so that it is expressed in terms of lower probabilities (Walley, 1982) of drawing a black or a white ball (denoted by $\underline{p}_b$ and $\underline{p}_w$, respectively):

$$\Delta'_A = \{(\underline{p}_b, \underline{p}_w) \in \mathbb{Q}^2 | \underline{p}_b + \underline{p}_w \leq 1\}$$

where $\underline{p}_b = \frac{b}{n}$ and $\underline{p}_w = \frac{w}{n}$.

The true probability of drawing a white ball p_w belongs to the interval $[\underline{p}_w, \overline{p}_w]$, where $\underline{p}_w$ denotes the lower probability of drawing a white ball while $\overline{p}_w$ denotes the upper probability of drawing a white ball. We assume that the set of priors C is the the set of all probabilities lying between the corresponding lower and upper probability.

2.1 λ-maxmin criterion

Let $\succsim$ satisfy the Ghirardato, Maccheroni, and Marinacci (2004) axioms. Then $f \succsim g$ if and only if:

$$(1-\lambda)\min_{p\in C}\mathbb{E}_p u(f) + \lambda \max_{p\in C}\mathbb{E}_p u(f) \geq (1-\lambda)\min_{p\in C}\mathbb{E}_p u(g) + \lambda \max_{p\in C}\mathbb{E}_p u(g)$$

where $\lambda \in [0,1]$ and C the set of priors are uniquely defined and u is unique up to a strictly positive affine transformation.

Several examples:

1. Let $f := (x_u, W_{(0,0)}; x_l, B_{(0,0)})$, where x_u, x_l, such that $x_u > x_l$ be a winning and a losing prize, respectively. Such a prospect corresponds to complete ignorance (out of n balls no color is known). Then the probability of drawing a white ball p_w is between 0 (the lower probability) and 1 (the upper probability). Hence the representation functional becomes:

$$(1-\lambda)\min_{p_w\in[0,1]}\mathbb{E}u(f) + \lambda \max_{p_w\in[0,1]}\mathbb{E}u(f) = (1-\lambda)u(x_l) + \lambda u(x_u)$$

which is the Hurwicz rule.

2. Let $f := (x_u, W_{(b,w)}; x_l, B_{(b,w)})$, where x_l, x_u, $x_u > x_l$, are the winning and a loosing prize, respectively, and $b+w < n$. Such a prospect corresponds to partial ignorance (out of n balls the color of fewer than n balls is known). Then the probability of drawing a white ball p_w is between $\frac{w}{n}$ (the lower probability) and $\frac{n-b}{n}$ (the upper probability). W.l.o.g. we assume that $u(x_l) = 0$. Hence the representation functional becomes:

$$\begin{aligned}
&(1-\lambda)\min_{p_w\in\left[\frac{w}{n},\frac{n-b}{n}\right]}\mathbb{E}u(f) + \lambda \max_{p_w\in\left[\frac{w}{n},\frac{n-b}{n}\right]}\mathbb{E}u(f) \\
&= \left((1-\lambda)\tfrac{w}{n} + \lambda\tfrac{n-b}{n}\right)u(x_u) + \left((1-\lambda)\tfrac{n-w}{n} + \lambda\tfrac{b}{n}\right)u(x_l) \\
&= \Big((1-\lambda)\underline{p}_w + \lambda\overline{p}_w\Big)u(x_u)
\end{aligned}$$

3. Let $f := (x_u, W_{(b,w)}; x_l, B_{(b,w)})$, where x_l, x_u, $x_u > x_l$, are the winning and a loosing prize, respectively, and $b + w = n$. Such a prospect corresponds to the situation of risk (no uncertainty) since the color of all balls in the urn is known. Then the probability of drawing a white ball p_w equals $\frac{w}{n}$. The representation functional then becomes:

$$\mathbb{E}u(f) = \tfrac{w}{n}u(x_u) + \tfrac{n-w}{n}u(x_l)$$
$$p_w u(x_u) + (1 - p_w)u(x_l)$$

which is the vNM Expected Utility.

3 Ghirardato, Maccheroni, Marinacci (2004) with set of priors in the core and Subjective Expected Utility.

Let $f := (x_1, E_1; ...; x_n, E_n) \in F$, where $x_i < x_j$, for $i < j$, $i, j \in \{1, ..., n\}$. Let $p_f : X \to [0, 1]$ be the induced probability distribution of f, i.e. $p_f(x) = \sum_{s \in S, f(s)=x} p(s)$. This induced probability distribution is imperfectly known.

Proposition 3.1. *Let the set of probabilities C be given by*

$$C := \{p \in [0, 1]^n : \ \underline{p}_i \leq p_i \leq \overline{p}_i, \ i \in \{1, ..., n\}\}$$

where $\underline{p}_i$ is the lower probability of E_i and $\overline{p}_i$ is the upper probability of E_i. Then the GMM preference functional becomes:

$$(1 - \lambda) \min_{p \in C} \mathbb{E}_p u(f) + \lambda \max_{p \in C} \mathbb{E}_p u(f) = \mathbb{E}_\pi u(f)$$

where $\pi : \mathcal{S} \to [0, 1]$ is the subjective probability defined as follows:

$$\pi(E_1) = \underline{p}_1 + (1 - \lambda) \left(1 - \sum_i \underline{p}_i\right)$$
$$\pi(E_i) = \underline{p}_i, \quad i \in \{2, ..., n - 1\}$$
$$\pi(E_n) = \underline{p}_n + \lambda \left(1 - \sum_i \underline{p}_i\right)$$

3.1 General formulation

We now proceed to a more general formulation with the set of probabilities C being the core of a totally monotone capacity.

Let S be a finite state space and $\mathcal{S}$ be an algebra of subsets of S, called events. A function $v : \mathcal{S} \to [0, 1]$ is a **capacity** if it satisfies the following three conditions:

a) $v(\emptyset) = 0$,

b) $v(S) = 1$,

c) $A \subseteq B \Rightarrow v(A) \leq v(B)$, for $A, B \in \mathcal{S}$.

Condition c) is referred to as monotonicity (wrt to set inclusion). We will henceforth assume that the capacity v is defined on the algebra $\mathcal{S}$ and consider only the events that belong to $\mathcal{S}$.

A capacity on $\mathcal{S}$ is **k-monotone** ($k \geq 2$) if for any family of k-subsets $A_1, ..., A_k$ of S, it holds:

$$v(\cup_{i \in K} A_i) \geq \sum_{I \subseteq K, I \neq \emptyset} (-1)^{|I|+1} v(\cap_{i \in I} A_i),$$

where $K := \{1, ..., k\}$. A capacity is **totally monotone** if it is k-monotone for every $k \geq 2$.

Möbius transform of v is a function $m : \mathcal{S} \to \mathbb{R}$ defined by:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} v(B), \quad A \subseteq S.$$

Conversely, Möbius transform m uniquely characterizes v because:

$$v(A) = \sum_{B \subseteq A} m(B), \quad A \subseteq S.$$

A Möbius transform m satisfies: $m(\emptyset) = 0$, $\sum\limits_{A \subseteq S} m(A) = 1$. If it also satisfies $m(A) \geq 0$, for $A \subseteq S$, then it is called a **mass allocation function**. Given a mass allocation function m, the corresponding unique capacity is called a **belief function** b.

Shafer (1976) has shown that a capacity is totally monotone if and only if it is a belief function.

Let $f := (x_1, E_1; ...; x_n, E_n) \in F$, where $x_i < x_j$, for $i < j$, $i, j \in \{1, ..., n\}$. To avoid unnecessary complication we assume that $\mathcal{S}$ is an algebra induced by the partition $\{E_1, ..., E_n\}$ of S. We assume that v is a totally monotone capacity on $\mathcal{S}$. It represents the decision maker's knowledge about $\mathcal{S}$. Let $\text{Core}(v)$ denote the set of all probability measures consistent with the decision maker's knowledge v. It is defined as a probability measure p such that $p(A) \geq v(A)$, for all $A \subset S$ and $p(S) = v(S)$. For any $E \in \mathcal{S}$, we define $\underline{p}(E) = v(E) = m(E)$ as a lower probability of E. For any $S \in \mathcal{S}$, we define $\overline{p}(E) = \sum_{A \supseteq E} m(A)$ as the upper probability of E. The lower/upper probability of E is interpreted as the lowest/highest possible probability value that can be assigned to the event E that is consistent with the decision maker's knowledge v. Let $B_i := E_1 \cup ... \cup E_i$, $G_i := E_i \cup ... \cup E_n$, for $i = 1, ..., n$ denote the "get at least as Bad as x_i" and "get at least as Good as x_i" events, respectively.

Proposition 3.2. *For a totally monotone capacity v, the following holds:*

$$\min_{p \in \text{Core}(v)} \mathbb{E}_p u(f) = \mathbb{E}_\pi u(f)$$

where $\pi(E_i) = v(G_i) - v(G_{i+1})$, *for* $i \in \{1, ..., n-1\}$, $\pi(E_n) = v(G_n) = v(E_n)$.

Proof. Since we are minimizing $\mathbb{E}_p u(f)$ over probability measures $p \in \text{Core}(v)$, we want to assign the highest possible probability in the core of v to the event E_1 with the lowest prize x_1. Hence

$$\begin{aligned}
\pi(E_1) = \overline{p}(E_1) &= \sum_{A \supseteq E_1} m(A) \\
&= \sum_{A \subseteq G_1} m(A) - \sum_{A \subseteq G_2} m(A) \\
&= 1 - v(G_2)
\end{aligned}$$

Out of the remaining $1 - \pi(E_1) = \sum_{A \subseteq G_2} m(A)$ probability mass we assign as much as possible to the event E_2. Continuing this way to each event E_i, $i \in \{1, ..., n-1\}$ we assign:

$$\begin{aligned}
\pi(E_i) &= \sum_{E_i \subseteq A \subseteq G_i} m(A) \\
&= \sum_{A \subseteq G_i} m(A) - \sum_{A \subseteq G_{i+1}} m(A) \\
&= v(G_i) - v(G_{i+1})
\end{aligned}$$

Finally, to the best event E_n we assign the lower probability of E_n:

$$\begin{aligned}
\pi(E_n) &= \sum_{E_n \subseteq A \subseteq G_n} m(A) \\
&= v(G_n) = v(E_n) = \underline{p}(E_n)
\end{aligned}$$

□

Proposition 3.3. *For a totally monotone capacity v, the following holds:*

$$\max_{p \in \text{Core}(v)} \mathbb{E}_p u(f) = \mathbb{E}_{\pi'} u(f)$$

where $\pi'(E_i) = v(B_i) - v(B_{i-1})$, *for* $i \in \{2, ..., n\}$, $\pi'(E_1) = v(B_1) = v(E_1)$

Proof. The proof is very similar to the proof of Proposition 3.2. We assign probability mass recursively in the following way: assign the highest possible probability in the core of v to the event E_n with the highest prize x_n:

$$\begin{aligned}\pi'(E_n) = \overline{p}(E_n) &= \sum_{A \supseteq E_n} m(A) \\ &= \sum_{A \subseteq B_n} m(A) - \sum_{A \subseteq B_{n-1}} m(A) \\ &= 1 - v(B_{n-1})\end{aligned}$$

Out of the remaining $1 - \pi'(E_n) = \sum_{A \subseteq B_{n-1}} m(A)$ probability mass we assign as much as possible to the event E_{n-1}. Continuing this way to each event E_i, $i \in \{2, ..., n\}$ we assign:

$$\begin{aligned}\pi'(E_i) &= \sum_{E_i \subseteq A \subseteq B_i} m(A) \\ &= \sum_{A \subseteq B_i} m(A) - \sum_{A \subseteq B_{i-1}} m(A) \\ &= v(B_i) - v(B_{i-1})\end{aligned}$$

Finally, to the worst event E_1 we assign the lower probability of E_1:

$$\begin{aligned}\pi'(E_1) &= \sum_{E_1 \subseteq A \subseteq B_1} m(A) \\ &= v(B_1) = v(E_1) = \underline{p}(E_1)\end{aligned}$$

□

Proposition 3.4. *For a totally monotone capacity v and $\lambda \in [0, 1]$, the following holds:*

$$\lambda \min_{p \in \mathrm{Core}(v)} \mathbb{E}_p u(f) + (1 - \lambda) \max_{p \in \mathrm{Core}(v)} \mathbb{E}_p u(f) = \mathbb{E}_{\pi''} u(f)$$

where $\pi''(E_i) = \pi'(E_i) + \lambda \left[\sum_{\substack{A \cap B_i \neq \emptyset \\ A \cap B_i^c \neq \emptyset}} m(A) - \sum_{\substack{A \cap B_{i-1} \neq \emptyset \\ A \cap B_{i-1}^c \neq \emptyset}} m(A) \right]$, *and m is a mass allocation function associated with the capacity v.*

Proof. Observe that $B_i = G_{i+1}^c$, for $i = \{1, ..., n-1\}$. Observe further that for any partition $\{B_i, B_i^c\}$, $i = \{1, ..., n\}$, we can decompose all the probability mass into three parts:

$$1 = \sum_{A \subseteq B_i} m(A) + \sum_{A \subseteq B_i^c} m(A) + \sum_{\substack{A \cap B_i \neq \emptyset \\ A \cap B_i^c \neq \emptyset}} m(A)$$

Hence for E_i, $i = \{1, ..., n\}$ the value $\pi''(E_i)$ equals:

$$\begin{aligned}
\pi''(E_i) =& \lambda\pi(E_i) + (1-\lambda)\pi'(E_i) \\
=& v(B_i) - v(B_{i-1}) \\
&+ \lambda\left[v(B_{i-1}) + v(B_{i-1}^c) - v(B_i) - v(B_i^c)\right] \\
=& \pi'(E_i) + \lambda\left[\sum_{\substack{A\cap B_i\neq\emptyset \\ A\cap B_i^c\neq\emptyset}} m(A) - \sum_{\substack{A\cap B_{i-1}\neq\emptyset \\ A\cap B_{i-1}^c\neq\emptyset}} m(A)\right]
\end{aligned}$$

□